## WHAT YOUR COLLEAGUES ARE SAYING . . .

"This is an amazing resource for anyone who wonders, 'why?' Mathematician James Tanton takes us into an exploration of mathematical conundrums and curiosities like dividing by zero, multiplying two negatives, and fraction division, showing us why they work and giving us creative new ideas for helping learners. It is a must-read for all math educators and the mathematically curious."

**Jo Boaler**
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Stanford, CA

"Math often feels like a play we've seen too many times—we know every twist and turn. Or so we thought. In *Math Decoded*, James Tanton takes us backstage, revealing surprising depth in even the most familiar concepts, and shows us not only new ways to understand mathematics, but also new ways to teach it more effectively. This is a backstage tour you don't want to miss."

**Peter Liljedahl**
Professor, Mathematics Education, Simon Fraser University
Author, *Building Thinking Classrooms in Mathematics*
Vancouver, British Columbia, Canada

"James Tanton is a gift to mathematics education. There are few people who I consistently learn so much from, and Tanton is one of them. This book is no exception. His playful and rigorous style makes important questions in K–12 math highly accessible. These are questions that are relevant whether you're highly experienced or teaching mathematics for the first time. I dare you to read it and not be wowed!"

**Michaela Epstein**
Founder & Director, Maths Teacher Circles
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"*Math Decoded* is James Tanton at his brilliant best: playful, profound, and passionately human. I'm constantly inspired by how he makes the abstract feel alive, and the complex feel inviting. This book isn't just a masterclass in mathematical insight; it's a love letter to the 'why' that fuels every curious learner."

**Eddie Woo**
Professor of Practice, Mathematics Education, University of Sydney
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"James Tanton is a master at making complex math ideas easy to understand as readers are presented with concrete examples that extend to deep, mathematical ideas. Following each big idea, activities are provided to engage your students in topics they (and sometimes teachers) often find confusing. This book is a worthwhile read for any educator."

**Abigail Bates**
Math Teacher, Bear Creek High School
Lodi Unified School District
Stockton, CA

"This is a brilliant resource from a brilliant math educator who excels at unpacking complex mathematical concepts into simple, accessible ideas for everybody. Tanton brings meaning, clarity, and joy to mathematics. The book masterfully addresses 'why' questions, providing profound insights into numbers and their operations. A must-read for every educator!"

**Jennifer Chang Wathall**
Education Consultant, JCW Consulting
Hong Kong

"Tanton brings to mathematics a quirky 'what?,' 'why?,' 'how?' approach that never fails to arouse curiosity. He's grabbed my attention on issues I thought I knew inside-out. For a school-aged audience, his approach can be dynamite. This book is a wonderful, inspiring resource of intriguing questions that bring 'everyday' math to life."

**Keith Devlin**
Stanford University Emeritus
Petaluma, CA

"A delightful journey into the playful depths of arithmetic. Tanton makes even classic stumpers—like 'why can't you divide by zero?' and 'what is infinity?'—feel like wondrous invitations to explore. This book is perfect for anyone who finds joy in the quirks and logic of mathematics."

**Scott Baldridge**
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"In classic James Tanton fashion, this book takes readers through some of mathematics' most perplexing questions. James masterfully tackles the timeless student query—'*Why do we need to do this?*'—and transforms it into an invitation to wonder, discovery, and delight. With his signature joy and clarity, he helps readers make sense of the very problems that once felt out of reach. A much-needed resource that brings the beauty and purpose of mathematics to life!"

**Sue Looney**
Founder, Looney Math Consulting
North Easton, MA

"As math teachers, how many times do we hear '*Why?*' from our students? I have heard it too many times to count and have been frustrated when I didn't have the answer. Not anymore! In *Math Decoded*, James Tanton takes the reader through the story of mathematics that illustrates the power and beauty behind the numbers. In this book, math transforms from stand-alone rules to intertwining concepts. I highly recommend this book to all teachers looking to deepen their own knowledge of mathematics and that of their students."

**Robin Kubasiak**
Math Teacher and Department Head, Sturgis High School
Bronson, MI

"This book is a fast, fun, and inspiring journey through the big questions in learning and teaching mathematics! It sparks curiosity, joy, and deep understanding by offering clear insights, hands-on examples, and powerful ways to make connections to big ideas we once struggled to explain. A valuable and uplifting resource for students and teachers alike!"

**Alicia Burdess**
Numeracy Lead Teacher, Grande Prairie and District Catholic Schools
Grande Prairie, Alberta, Canada

"Reading this book, I could feel Tanton's signature joy and enthusiasm for taking big ideas and breaking them into bite-size pieces that are easy to digest, while also blowing your mind. This is an amazing resource for any teacher looking to better understand, as Tanton says, 'the poetry of mathematics.' The journey through the book was both incredibly validating and completely inspiring. Each idea sparks thoughts about classroom conversations my students have engaged in and how to take these conversations further with the book's tasks."

**Casey McCormick**
5th–8th Grade Math Teacher
Citrus Heights, CA

"James Tanton makes arithmetic a playground of wonder and delight, and he shows how you can too. Tanton's insights shine as he illuminates the creative possibilities of math when you allow numbers to take a life of their own in wider contexts. His tantalizing philosophical questions will broaden your thinking and get students talking. *Math Decoded* is both a satisfying read and practical resource for every teacher and parent to explain why numbers behave as they do."

**Francis Su**
Author, *Mathematics for Human Flourishing*
Pasadena, CA

"What a wonderful and unique perspective James Tanton brings to this delightful book! His approach is irresistible: You'll come away with a richer and more profound understanding of numbers and mathematical thinking."

**Dan Finkel**
Founder, Math for Love
Seattle, WA

"To know James Tanton is to know that mathematics is a fully joyful and human endeavor! This book showcases his perspective perfectly through developing critical mathematical ideas in ways that just make sense. Tanton's innovative approach provides educators with an untangled and transformational rethinking of mathematics that is fun to read and accessible for all."

**April Strom**
Mathematics Faculty, Chandler-Gilbert Community College
Chandler, AZ

"Powerful ways of thinking about numbers and operations are key to understanding and enjoying mathematics, but too often those foundations are seen as disconnected and joyless. In *Math Decoded*, James Tanton weaves a delightful storyline that connects, makes sense of, and brings joy to the study of mathematics. This is a book for everyone, and along the way, you'll have many opportunities to say to yourself 'I never thought of it that way, but I wish I had.'"

**Ted Coe**
Founder, Coequal Mathematics LLC
Scottsdale, AZ

"With his Aussie wit, Tanton takes us on an almost whimsical journey to help us see a story of mathematics and think deeply about the inherent connections that exist. Rather than giving an instruction on how to teach in the classroom, he invites us to take a look back on many intriguing parts of school mathematics."

**Pam Harris**
CEO, Math Is Figure-Out-Able
Author, Developing Mathematical Reasoning
Kyle, TX

# Math Decoded

## Explaining the Why Behind the Numbers and How They Operate

James Tanton

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Visit the companion website at
**https://companion.corwin.com/courses/mathdecoded**
for downloadable resources.

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## VIDEO SERIES: FRACTIONS—THE MATH AND NOTHING BUT THE MATH

**Note From the Publisher:** The author has provided video and web content throughout the book that is available to you through QR (quick response) codes. To read a QR code, you must have a smartphone or tablet with a camera. We recommend that you download a QR code reader app that is made specifically for your phone or tablet brand.

Videos may also be accessed at
**https://companion.corwin.com/courses/mathdecoded**

# ACKNOWLEDGMENTS

First and foremost, my deepest gratitude to the many individuals who have helped develop and shape my understanding of mathematics: the colleagues, mentors, teachers, and students I have had the privilege to work with and learn from over the years. Your insights, curiosity, and questions have been gifts.

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My heartfelt thanks to the Corwin team for taking on a book project that is somewhat "outside the norm" and helping me transform a rough manuscript into something worth taking on. I am especially grateful to Associate Content Director, Mathematics, Erin Null for her guidance and expertise throughout this process.

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# ABOUT THE AUTHOR

**James Tanton,** born in Adelaide, Australia, earned his PhD in mathematics from Princeton University in 1994. A passionate ambassador for mathematics, James has dedicated his career to sharing the joy, wonder, and creativity that mathematics inspires. With more than 20 years of teaching experience in both university and high school classrooms, he now works globally as an independent consultant, bringing his unique perspective to learners of all ages.

James is an author, video creator, and speaker who specializes in making mathematics accessible and delightful. He collaborates with teachers, designs innovative curricula, and leads professional development sessions and demonstration classes around the world. As the cofounder of the Global Math Project, James is on a mission to transform the way people perceive mathematics. His initiative has reached more than 7 million students and educators across more than 160 countries, sparking curiosity and genuine excitement about school mathematics.

James currently resides in Berkeley, California, with his wife, Lindy Elkins-Tanton.

Learn more about bringing James Tanton to your school or district at www.jamestantonmath.com

*To all the educators of mathematics, who strive to bring meaning, clarity, humanity, insight, and audacious joy into their classrooms.*

# INTRODUCTION

## THE WHY OF THIS BOOK

For a long time, mathematics education has emphasized the grammar of math—the rules, procedures, and formulas that lead to correct answers, often with speed as the primary measure of mastery. Over the past two decades, however, there has been a growing shift toward a model of mathematics education that prioritizes deep thinking, problem-solving, and making connections. Yet, this shift has been implemented inconsistently and unevenly. Just as English language arts embraces both the grammar and the poetry of language, mathematics education can celebrate both the mechanics and the beauty of mathematics.

This book is about the poetry of mathematics. It delves into the joy of discovering structure and form, the thrill of uncovering insights, and the profound satisfaction of seeing how the playful exploration of ideas illuminates and enhances our understanding of real-world phenomena. It invites readers to see mathematics not just as a practical tool, but as a deeply human story—a story of curiosity, creativity, and wonder.

Moreover, this book is designed to deepen your own mathematical knowledge, while equipping you to address some of those tricky "but, why" questions that often arise in the classroom:

- Why is a negative times a negative positive?
- Why can't you divide by zero?
- Does 0.999 . . . equal 1, or does it not?
- Is zero positive, negative, both, or neither?
- Why does "Keep-Change-Flip" work for dividing fractions?

Students are naturally curious. They wonder, they question, and they crave understanding of the why. Yet, by and large, we've encouraged them to view school mathematics as purely grammar with no place to question. Let's invite them into the poetry as well by embracing the why. This book shows how.

By revealing the deeper structure of mathematics, this book provides the context and insights needed to make sense of your classroom curriculum. It helps you understand its choices and responses—or lack thereof—to those tricky "why" questions, while giving you a solid foundation to navigate these moments as a natural part of your day-to-day teaching.

## WHO THIS BOOK IS FOR

This is not a textbook. It is a resource for personal inspiration, classroom enrichment, and professional growth—designed for educators and anyone curious about the deeper truths of mathematics. You don't need to change your teaching practices to benefit from this book. Instead, it provides ways to deepen your understanding of the mathematics behind the scenes of your teaching materials, enriching your conversations with students and helping you confidently address their most challenging questions. To further support this, I've created a series of videos to complement this book that give a sense of the style and approach and delivery you could take in answering such questions.

Educators at all levels—upper elementary, middle school, high school, and beyond—will find value in this resource. You can engage with it individually, or as part of a group or book study. You may enjoy trying out some of the tasks with colleagues in your professional learning community or integrating this book into your professional development efforts.

This book is designed not only to enrich your students and their learning experience but also to inspire and empower *you*.

## THE STORY BEHIND THE STORY

Mathematics is a remarkable discipline—rooted in real-world scenarios yet capable of transcending them entirely. It is both practical and profound, grounded in the everyday while reaching far beyond it. This dual nature makes mathematics a uniquely challenging and fascinating subject to teach and learn.

It is natural to want to fit mathematics into tidy real-world frameworks. This often leads to some definitive statements:

- Subtraction is "taking away."
- Multiplication is "repeated addition."
- Division is "repeated subtraction."
- A fraction is "an answer to an equal-sharing problem."
- Exponentiation is "repeated multiplication."

While these explanations are helpful starting points, they can be reductive and limiting. They may cause intellectual discomfort as students (and even educators) encounter mathematical ideas that stretch beyond these definitions. For instance:

- How do you make sense of $3 - 5$ if subtraction is only “taking away”?
- If multiplication is “repeated addition,” what do $(-2) \times (-3)$ and $0 \times 0$ mean?
- Why does $2^0$ equal 1? And what about $0^0$?
- Is $\frac{2}{(-3)}$ a fraction? Is it even a number?

Mathematical topics are inspired by real-world situations—counting, classifying, dividing, sharing, restoring. Yet mathematics is always bigger and bolder than any single context, including those that originally motivated it!

This book delves into the space between these practical beginnings and the broader, more abstract world of mathematics. It is a mathematical narrative—a story chronicling the evolution of numbers and their arithmetic. Each chapter begins with real-world motivations and then elevates the discussion to unexpected heights, revealing the beauty, coherence, and creativity inherent in mathematics.

## WHAT’S INSIDE

Each chapter follows a consistent structure to guide readers through the exploration of mathematical ideas:

1. **Something to Mull On:** An intriguing question or task to spark curiosity and encourage deep thinking.
2. **Motivation From the Real World:** A discussion of the concept’s origins and its connections to everyday life.
3. **A Key Takeaway:** The core idea at the heart of the chapter.
4. **Being Bold With the Math:** An exploration of how the concept extends beyond its initial context, revealing its broader implications.
5. **More Mulling and Some Classroom Tasks:** Final reflections along with activities and tasks tailored for students at different grade levels.

## WHAT'S OUTSIDE(!)

As mentioned, we offer a suite of complementary resources for your enjoyment and support, available at https://companion.corwin.com/courses/mathdecoded:

- **Exclusive Support Videos:** A collection of videos created specifically to accompany this book, including a suite of 10 videos that present the complete and rigorous mathematics behind the scenes of fractions.
- **Answer Key:** Solutions to select problems presented in the book, offering guidance and clarity for deeper understanding.
- **Summary:** A complete list of key takeaways from each chapter, providing an at-a-glance view of how the story of numbers and their arithmetic unfolds.
- **Supplemental Essays for Further Exploration:**
  - Follow-on material for Chapter 1: *"There is more than one type of infinity."*
  - A paper-folding proof following Chapter 20: *"The square root of two is irrational."*

Visit the companion website to access these resources and deepen your engagement with the book's content!

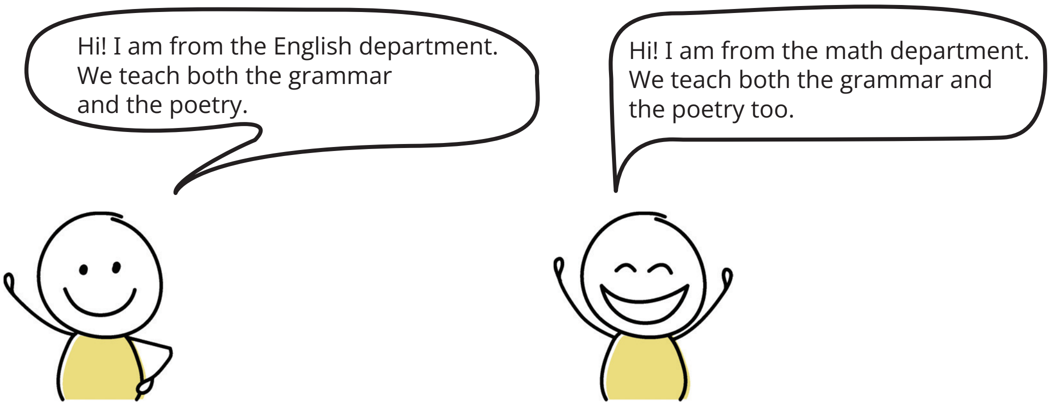
## AN INVITATION

Join me on this journey as we uncover the bold, beautiful, and deeply human nature of mathematics (see Figure 0.1). Let's together explore the vast and varied landscape of numbers, tackling questions that have challenged and inspired mathematicians for centuries. Along the way, I hope to rekindle your sense of wonder and curiosity—and perhaps inspire you to share that wonder with your students.

Mathematics is both a practical tool and a transcendent art form. Let's embrace its beautiful structure, celebrate our shared human struggles to uncover it, and revel in its stunning poetry.

Welcome to the story of numbers.

**Figure 0.1** • *The Grammar and Poetry of Math*




Source: iStock.com/Karolina Madej

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*(Continued)*

(Continued)

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## CHAPTER 1

# COUNTING

### Something to Mull On

*Have you ever noticed something curious about the way we name numbers in English?*

We have 10 basic names—*zero*, *one*, *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, and *nine*—for the 10 Hindu–Arabic numerals (0 through 9). But English gives special names to the next two numbers as well—*eleven* and *twelve*. It then falls into a naming pattern for the subsequent numbers, *thirteen* for "three and ten," *fourteen* for "four and ten," and so on, up to *nineteen*, but changes that pattern the next number onward. We have *twenty* (for "two tens"), followed by *twenty-one*, *twenty-two*, and so forth, then *thirty* (for "three tens"), *thirty-one*, and so on.

| | | |
|---|---|---|
| zero | thirteen | twenty |
| one | fourteen | twenty-one |
| two | fifteen | twenty-two |
| | | ⋮ |
| three | sixteen | thirty |
| four | seventeen | thirty-one |
| five | eighteen | ⋮ |
| six | nineteen | forty |
| seven | | ⋮ |
| eight | | fifty |
| nine | | ⋮ |
| ten | | sixty |
| eleven | | ⋮ |
| twelve | | |

Even though our number system is based on "ten-ness"—ones, tens, hundreds (10 tens), thousands (10 hundreds), and so on—English number names seem to emphasize a sense of "twelve-ness" and "twenty-ness" as well. Why is that?

A possible clue lies in our dual use of the word *digit*: It refers not only to any numeral from 0 to 9 but also to our fingers, thumbs, and toes (Figures 1.1 and 1.2). Could the connection between counting and our physical digits explain this linguistic curiosity?

**Figure 1.1** • *Digits Shown Symbolically*

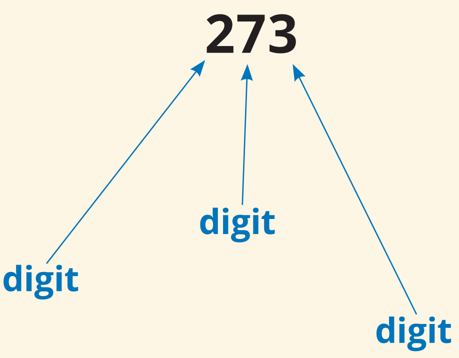
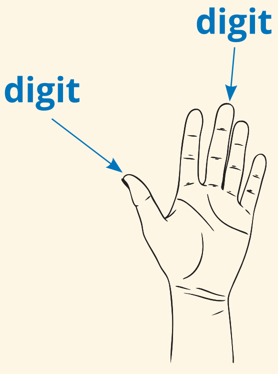



**Figure 1.2** • *Digits on a Hand*




Source: hand icon by iStock.com/Arkadivna

## COUNTING IN THE REAL WORLD

Counting is fundamental to human experience. We can't imagine living in a world without at least some counting: none, one, two, many!

Young children naturally delight in counting—ascending stairs, descending them, sometimes getting different numbers each time. Whether they ponder these differences or not, they're engaging with a sophisticated mental process. And counting is surprisingly complex. It involves memorizing a list of symbols and their names, as well as having a shared understanding of how to extend this list indefinitely.

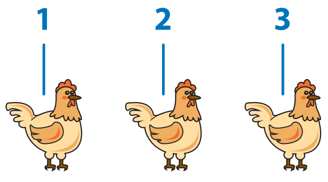
As a community, we've agreed on this list of symbols, which we say represents the **counting numbers**:

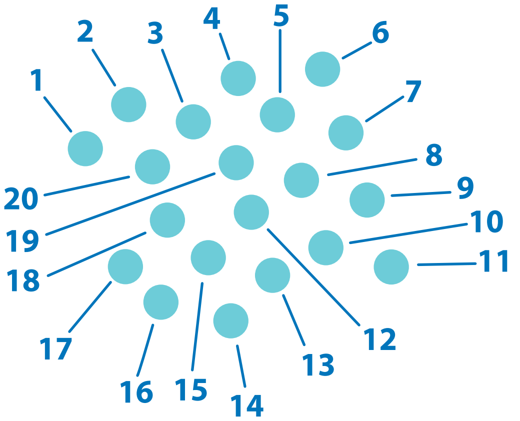
**1 2 3 4 5 6 7 8 9 10 11 12 . . .**

We know how to continue this list: After 12 (twelve) comes 13 (thirteen), after 99 (ninety-nine) comes 100 (one hundred), after 8,672,098,037 (I won't even attempt to say that one!) comes 8,672,098,038, and so on.

To **count** a set of objects, we assign each item a number from this sequence, starting with 1 (one). When we finish assigning numbers, the last one reached tells us how many objects are in the set (Figure 1.3).

**Figure 1.3** • *One-to-One Correspondences Between Counting Numbers and Objects*



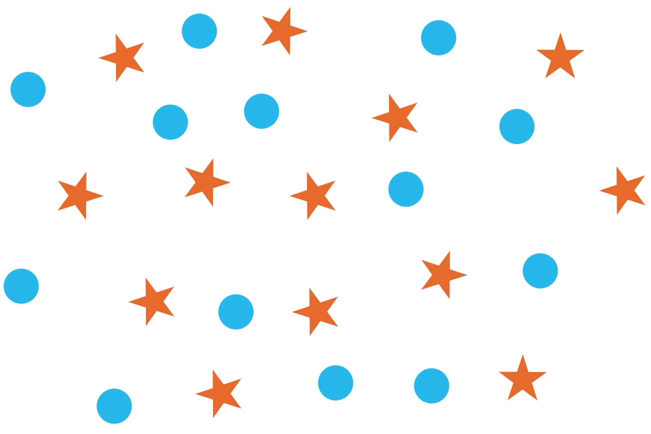




In the early grades, we encourage students to touch the objects as they count them, making the one-to-one correspondence between objects and number names literally tangible.

## A KEY TAKEAWAY

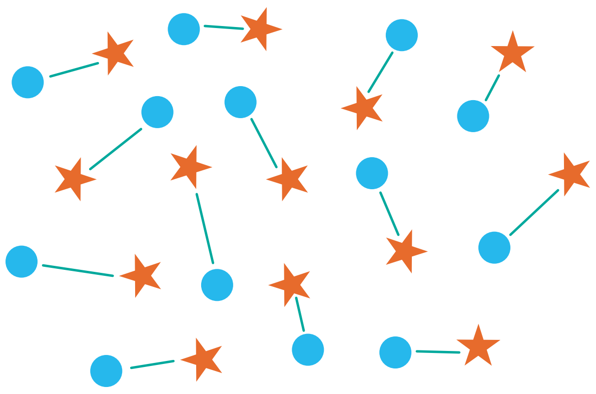
Drawing lines between objects in sets—I'll call them "leashes"—provides a way to recognize that two sets are the same size. For example, here's a collection of dots and stars (Figure 1.4). It's hard to tell if there is an equal count of each.

**Figure 1.4** • *A Collection of Dots and Stars*



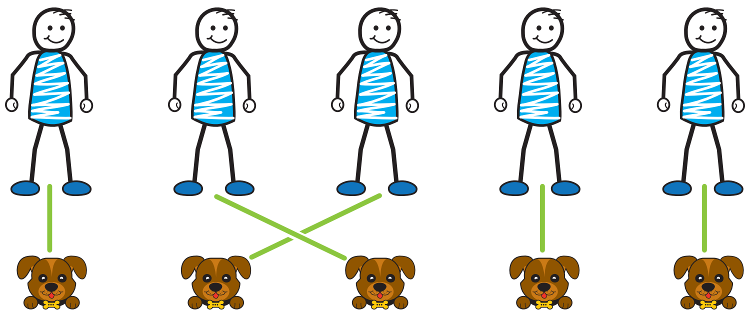
But, as Figure 1.5 makes clear, they are equal in number (without ever counting *13 of either shape*).

**Figure 1.5** • *Dots and Stars Leashed*



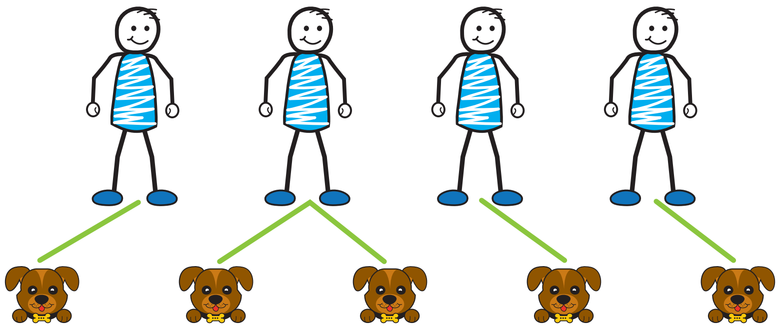
Leaning into this leashing idea, here's a picture of some people walking their dogs (Figure 1.6). Each dog is leashed to one person, and each person is leashed to one dog. Again, we see that the set of people and the set of dogs are each the same size.

**Figure 1.6** • *People Each Leashed to One Dog*



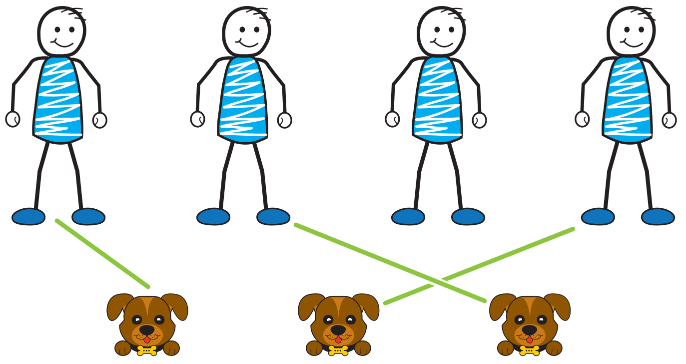
In Figure 1.7, one person is leashed to more than one dog. Our intuition says that the set of people and the set of dogs are not the same size in this case.

**Figure 1.7** • *Three People Each Leashed to One Dog, One Person Leashed to Two Dogs*



And we'd agree matters are also problematic if a person (or a dog) is skipped in a leashing pattern (Figure 1.8).

**Figure 1.8** • *A Person Skipped in Leashed Dogs*



From these observations, we can declare a key takeaway:

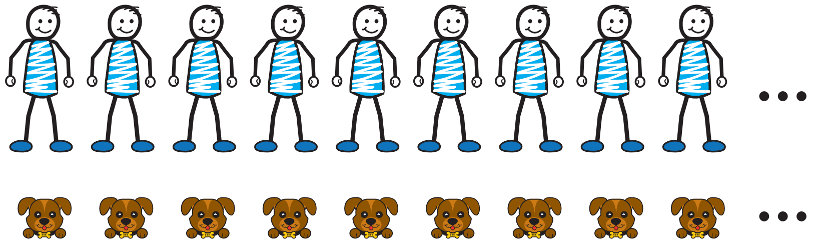
> Two sets of objects are the **same size** if it is possible to describe a leashing pattern between objects so that each item of the first set is leashed to exactly one item of the second set and each item of the second set is leashed to exactly one item of the first set.

We've created a concept of size that is entirely independent of how we choose to name numbers. This idea is profound because it demonstrates that the concept of "size" transcends counting or number names. It opens the door to comparing the sizes of sets that cannot be counted—infinite sets!

## BEING BOLD WITH THE MATH

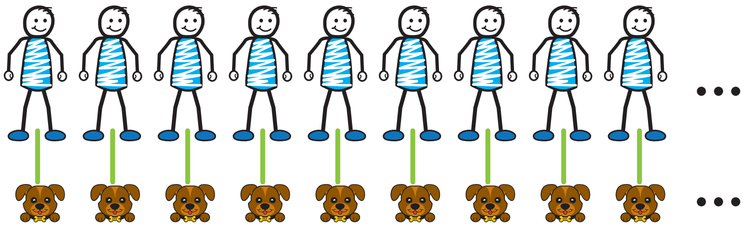
Let's be bold and extend this idea of leashing to sets we typically consider beyond counting. Here's an infinite line of people going infinitely far to the right and an infinite line of dogs also going infinitely far to the right (Figure 1.9). Are the set of people and the set of dogs in this picture "the same size"?

**Figure 1.9** • *Infinite Lines of Dogs and People*



The answer is yes if we follow this leashing idea. We can certainly envision a way to draw leashes so that each dog is leashed to a person and each person is leashed to a dog (Figure 1.10).

**Figure 1.10** • *A Leashing Pattern Between Two Infinitely Long Sets*

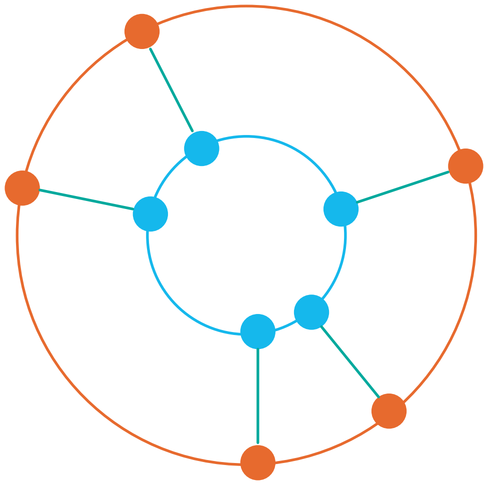


Even with the ellipses ( . . . ), we can see how this leashing pattern will continue: The tenth dog and the tenth person will be leashed together, the thousandth dog and the thousandth person will be leashed together, the ten-millionth dog and the ten-millionth person will be leashed together, and so on.

Describing, or just visually exhibiting, a leashing pattern that can be continued is enough to say that two different sets are the same size.

Four hundred years ago, Italian scientist and mathematician Galileo Galilei (1564–1642) observed that there are "just as many" points on a small circle as there are on a bigger circle. He imagined leashes between points on the circles (Figure 1.11).

**Figure 1.11** • *Corresponding Points on Two Circles*



He also observed that there seem to be just as many square numbers as there are numbers. For example, the number 1 can be leashed with $1 \times 1 = 1$, the number 2 with $2 \times 2 = 4$, the number 3 with $3 \times 3 = 9$, and so on (Figure 1.12).

**Figure 1.12** • *Leashed Square Numbers*

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | ... |
| \| | \| | \| | \| | \| | \| | |
| 1 | 4 | 9 | 16 | 25 | 36 | ... |

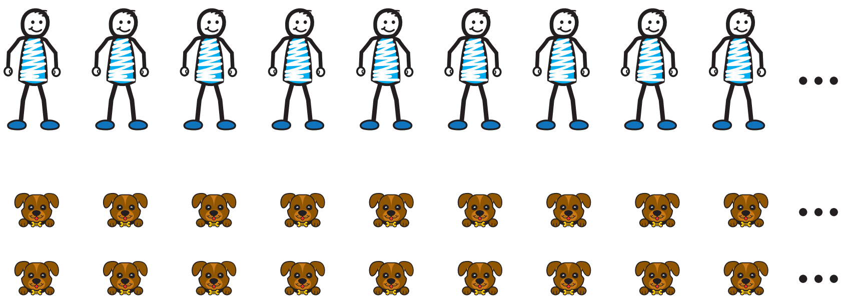
The set of square numbers is a subset of the counting numbers, but not the entire set. The fact that the set of counting numbers is the same size as one of its subsets (the set of square counting numbers) is possible only because the set of counting numbers is infinite.

In fact, this property can be taken as the definition of what it means for a set to be infinite:

> A set of objects is **infinite** if it is possible to appropriately leash the elements of the entire collection to one of its subsets.

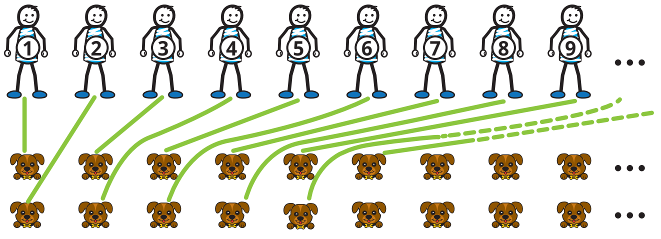
Two hundred years later, German mathematician Georg Cantor (1845–1918) took this play of infinite leash patterns to astounding heights. Here's an infinite row of people and two infinite rows of dogs (Figure 1.13).

**Figure 1.13** • *One Infinite Row of People and Two Infinite Rows of Dogs*



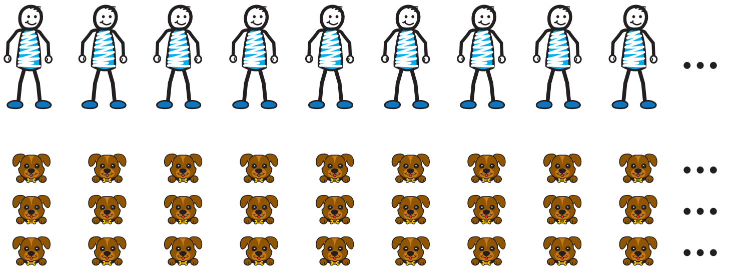
Are these two sets the same size? Or is the count of dogs "double the infinity" of the count of people? Surprisingly, leashing shows that there are just as many people as there are dogs! Here's a way to show a valid leashing pattern that can clearly be extended (Figure 1.14).

**Figure 1.14** • *An Extended Leashing Pattern*




If we number the people 1, 2, 3, 4, 5, and so on, we can match people 1, 3, 5, 7, and so on with the dogs in the first row and match people 2, 4, 6, 8, and so on with the dogs in the second row. Likewise, we can now show that a "triple infinity" of dogs is the same size as a "single infinity" of people (Figure 1.15).

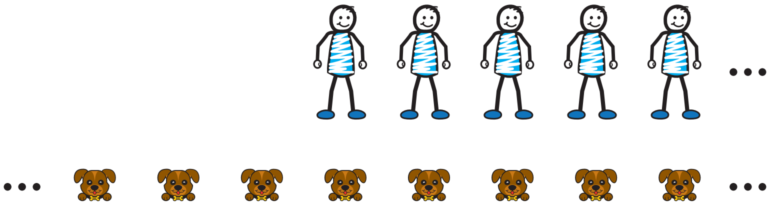
**Figure 1.15** • *A Triple Infinity of Dogs Is the Same Size as a Single Infinity of People*



How about a single infinity of people and a "double-ended infinity" of dogs?

Are these two sets the same size (Figure 1.16)?

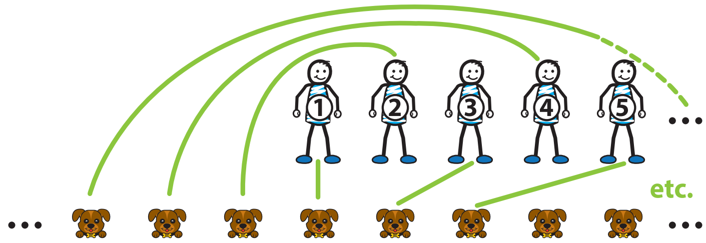
**Figure 1.16** • *A Single Infinity of People and a Double-Ended Infinity of Dogs*



What do you think? What does your instinct tell you? Is there a leashing pattern between people and dogs that works, or is there no possible pattern?

Again, perhaps surprisingly, there is a leashing pattern that shows a single infinity and a double-ended infinity being the same size. Number the people and match person number 1 to a dog, and then match people 3, 5, 7, 9, and so on to dogs to its right and people 2, 4, 6, 8, and so on to dogs to its left (Figure 1.17).

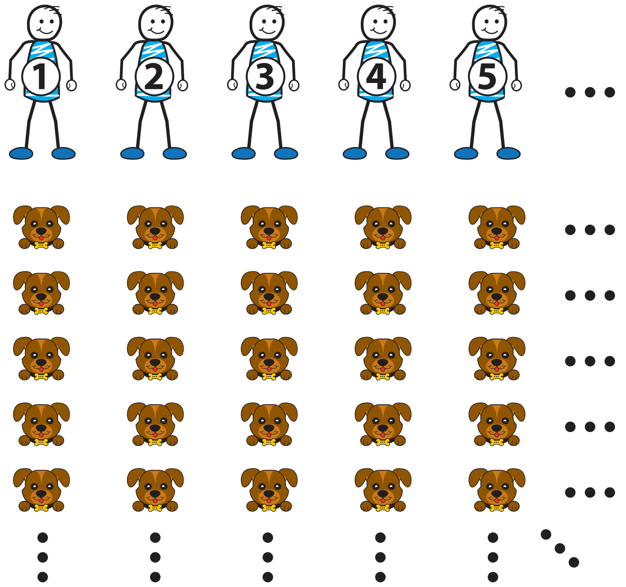
**Figure 1.17** • *Leashing a Single Infinity of People and a Double-Ended Infinity of Dogs*




We've now shown that double, triple, and double-ended infinitely big sets of dogs are all the same size as a single infinity of people numbered 1, 2, 3, 4, and so on. People call these infinite sets **countably infinite** because we are matching elements of those sets (the dogs) with the set of counting numbers (labeled people). All the infinite sets we've seen thus far are the same, countably infinite size. But surely a two-dimensional array of dogs—infinitely many rows of infinitely many dogs—is "more infinite" than a single countable infinity. There just can't be a leashing pattern between the people and dogs in this picture (Figure 1.18).

What do you think?

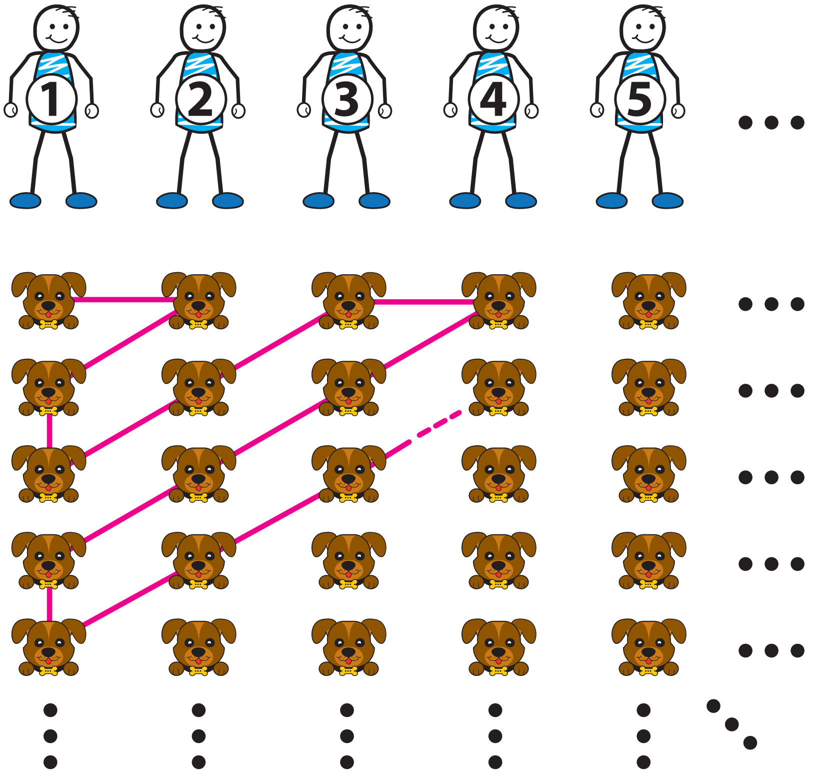
**Figure 1.18** • *Infinitely Many Rows of Infinitely Many Dogs*



You might be surprised to see that there is a leashing pattern that works!

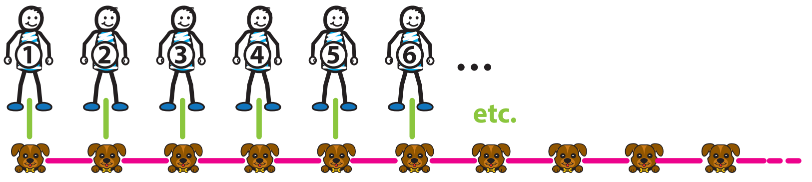
Start by drawing a zigzag line that weaves through the whole two-dimensional array of dogs (Figure 1.19).

**Figure 1.19** • *A Start to a Leashing Pattern for Infinitely Many Rows of Infinitely Many Dogs*



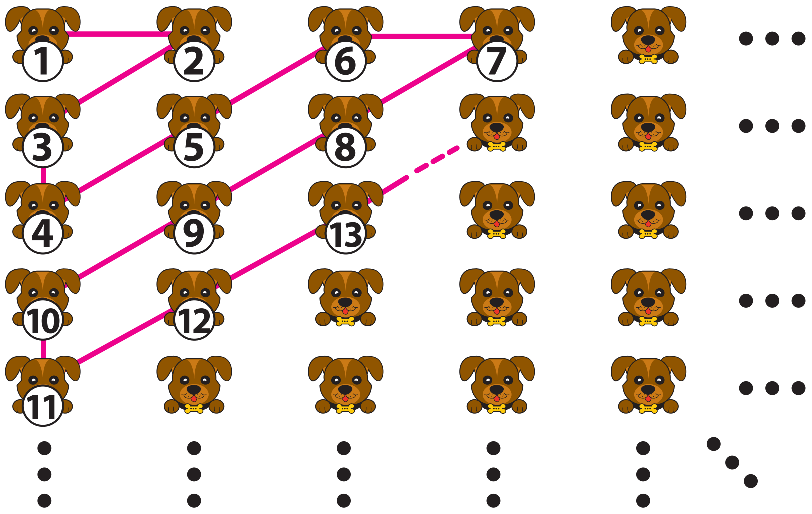

Then we can straighten out that line of dogs and display a leashing pattern (Figure 1.20).

**Figure 1.20** • *Straightening Out the Line of Dogs for the Leashing Pattern*



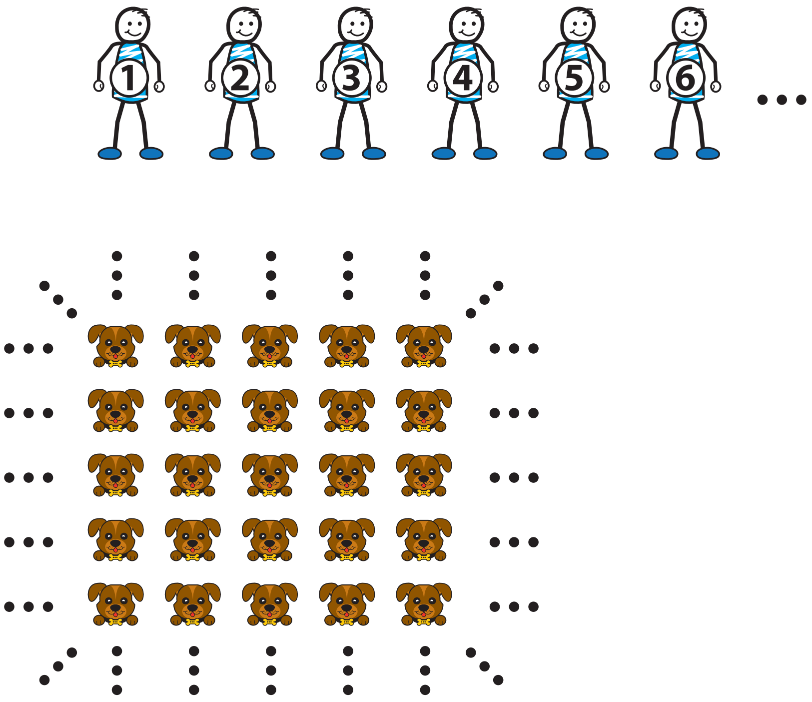

Actually, simply demonstrating a counting scheme—1, 2, 3, 4, and so on—that weaves its way through an infinite set without ever missing an element is enough to illustrate a leashing pattern. For example, Figure 1.21 tells us to which person each dog is leashed.

**Figure 1.21** • *Indicating to Which Person Each Dog Is Leashed*




Perhaps you can now envision a leashing pattern that shows that a full two-dimensional array of dogs is the same size as a single infinity of people. (Imagine a spiral pattern within Figure 1.22.)

**Figure 1.22** • *A Two-Dimensional Array of Dogs and a Single Infinity of People*




It seems that all the infinitely large sets we can imagine are the same size: "Infinity is infinity is infinity."

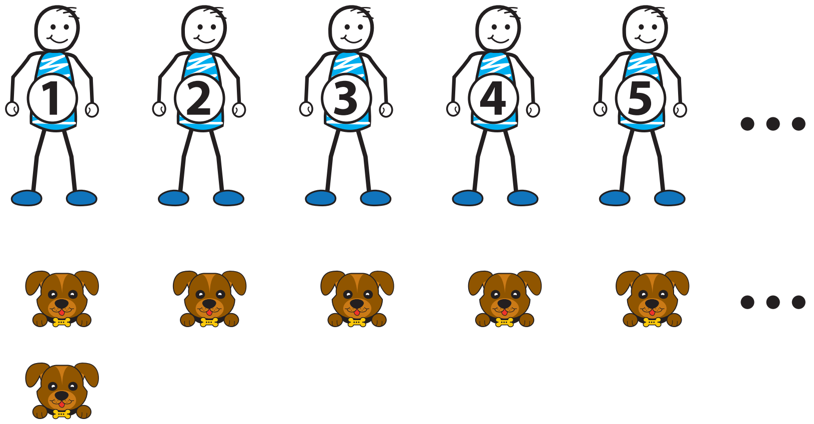
This makes the following schoolyard exchange moot.

**Kid A:** "I infinity dare you."

**Kid B:** "I infinity plus one dare you!"

Indeed, try proving that this set of people and this set of dogs are the same size (Figure 1.23).

**Figure 1.23** • *Are This Set of People and This Set of Dogs the Same Size?*




But before we risk falling into a sense of complacency with counting, consider how Cantor turned this notion on its head. He presented an example of a set that is truly more infinite than any set of people or dogs we've encountered so far. He proved that there is, in fact, more than one type of infinity!

If you're feeling mathematically adventurous, turn to the supplementary essay for Chapter 1 at https://companion.corwin.com/courses/mathdecoded for the details of this next bold turn.

## MORE MULLING AND SOME CLASSROOM TASKS

We humans were likely attracted to the number 10 for matters of counting and arithmetic because of our human anatomy: We have 10 digits on our hands, which naturally leads us to think in groups of 10. Check out some examples in Videos 1.1 and 1.2.

**Video 1.1 Weird Number Names**

qrs.ly/ihgt199

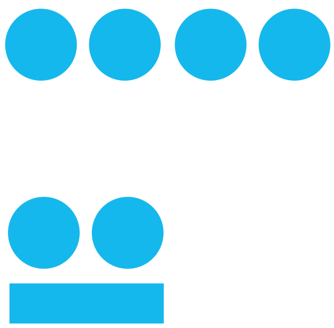
To read a QR code, you must have a smartphone or tablet with a camera. We recommend that you download a QR code reader app that is made specifically for your phone or tablet brand.

**Video 1.2 How Do You Count to Five on One Hand?**
qrs.ly/vqgt1dn

However, not all cultures developed a base-ten counting system. Some developed a base-twenty system. (Fingers and toes?)

The traditional counting system of the Welsh language expresses 87 as *pedwar ugain a saith*, which translates word for word to "four twenty and seven," to mean $4 \times 20 + 7$. The Maya of the Mayan civilization in Mesoamerica also used a base-twenty system. They wrote their numbers vertically—the count units at the bottom, the count of 20s above them, the count of $20 \times 20 = 400$s above those, and so on—using dots (for 1s) and bars (for 5s). Figure 1.24, for instance, shows "4" at the top and "7" below to be read as "four twenties and seven" to make 87.

**Figure 1.24** • *A Mayan Representation for 87*

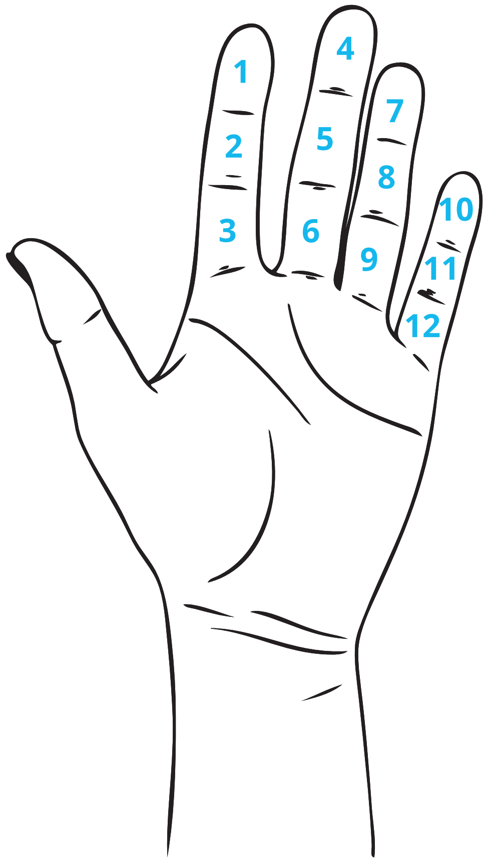


Remnants of base-twenty thinking can still be found in contemporary Western European culture. For example, Abraham Lincoln's famous Gettysburg Address begins, "Four score and seven years ago." The word *score* is an old term for "twenty," so Lincoln was saying "four twenties and seven years ago," which amounts to 87 years. Similarly, in French, the number 87 is expressed as *quatre-vingt-sept*, which translates literally to "four twenties seven."

It seems natural for humans to have developed base-ten and base-twenty number systems given our anatomy. However, some cultures have alternatively developed a base-twelve number system!

This could be because there is a straightforward way to count to 12 on one hand. We have four long digits, each naturally divided into three segments, along with a thumb that serves as a pointer (Figure 1.25). (In some parts of the world, particularly in India and Southeast Asia, it is still common for people to count to 12 on one hand.)

**Figure 1.25** • *Hand With Digits Numbered 1 to 12*




Source: hand icon by iStock.com/Arkadivna

Also, there is a practical reason for working with the number 12 in everyday life, which we still do to this day!

*How many items are in a dozen?* Answer: 12.

*How many inches are in a foot?* Answer: 12.

*How many hours are in a day—literally?* Answer: 12!

The first clocks constructed by humans were sundials, which only work during daylight hours. The ancient Egyptians divided the day into 10 main hours and 2 additional twilight hours—early morning and early evening. With the invention of water clocks and mechanical clocks, people could begin measuring time at night as well. Since the day was divided into 12 hours, they similarly divided the night into 12 hours.

The number 12 is also useful in matters of weights and measures. For example, one might want to purchase not a full unit of a quantity but perhaps only half, a third, or a quarter of that amount. These fractions are common in everyday situations, and working with groups of 12 makes handling these fractions much easier: Half, a third, and a quarter of a dozen are whole numbers, while this is the case for only half of 10.

Societies evolved embracing the concepts of ten-ness, twenty-ness, and twelve-ness simultaneously. As the English language evolved organically, too, it is perhaps not surprising it came to reflect these ideas as well.

## A TASK FOR ELEMENTARY SCHOOL GRADES

Encourage students to take a closer look at the names we use for numbers, and brainstorm new names that better reflect our base-ten system.

For instance, in the past, young students have proposed names like *onety* for 10, *onety-one* for 11, and so on. (This approach is similar to how numbers are named in Chinese, for instance, where 11 is expressed as *ten-one*, 12 as *ten-two*, and so forth.)

Students might also explore ways to make the spelling of number names more consistent. For example, while we write *sixty* and *seventy* for 60 and 70, the spellings for 20 (*twenty*), 30 (*thirty*), 40 (*forty*), and 50 (*fifty*) deviate from this pattern. What alternative spellings might create a more consistent system?

| | |
|---|---|
| zero | |
| one | |
| two | |
| three | |
| four | |
| five | |
| six | |
| seven | |
| eight | |
| nine | |
| ten | onety |
| eleven | onety-one |
| twelve | onety-two |

*(Continued)*

(Continued)

| | |
|---|---|
| thirteen | onety-three |
| fourteen | onety-four |
| fifteen | onety-five |
| sixteen | onety-six |
| seventeen | onety-seven |
| eighteen | onety-eight |
| nineteen | onety-nine |
| twenty | twoty |
| twenty-one | twoty-one |
| twenty-two | twoty-two |
| ⋮ | ⋮ |
| thirty | threety |
| ⋮ | ⋮ |
| forty | fourty |
| ⋮ | ⋮ |
| fifty | fivety |
| ⋮ | ⋮ |
| sixty | |
| ⋮ | |

While we can't change the naming conventions currently used by society, this type of focused activity helps students better understand and remember the linguistic quirks of our number names.

## A TASK FOR MIDDLE SCHOOL GRADES

Share this image of a number in a place-value chart with students. The chart is composed of 17 thousands, 17 hundreds, 17 tens, and 17 ones (Figure 1.26).

**Figure 1.26** • *Four 17s in a Place-Value Chart*

| 17 | 17 | 17 | 17 |
|---|---|---|---|
| **thousands** | **hundreds** | **tens** | **ones** |

Ask students to determine what number this really represents, and then to read their answer—18,887—slowly out loud to notice what they are saying.

Have students observe that even though the number is made up of 1 ten-thousand, 8 thousands, 8 hundreds, 8 tens, and 7 ones, we read it as "18 thousand" as though the first "digit" is 18 (Figure 1.27)! This is yet another quirk of English.

**Figure 1.27** • *18,887 in a Place Value Chart as It Is Spoken*

| | 18 | 8 | 8 | 7 |
|---|---|---|---|---|
| **ten-thousands** | **thousands** | **hundreds** | **tens** | **ones** |

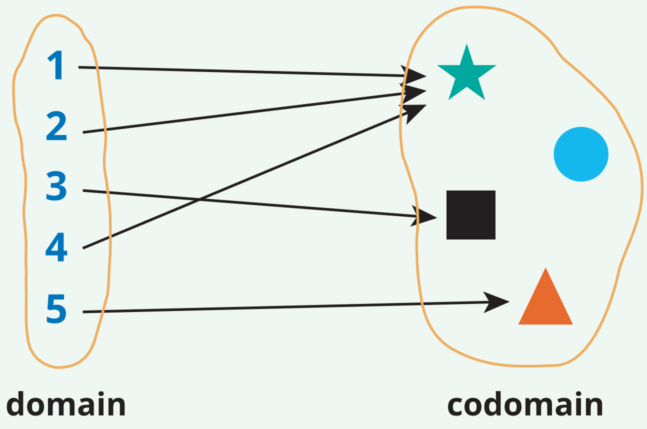
Encourage students to research how the number 18,887 is expressed in other languages. For example, share that in Korean this number is written as 일만 팔천 팔백 팔십 칠 (*il-man pal-cheon pal-baek pal-sip chil*), which translates literally as 1 ten-thousand, 8 thousands, 8 hundreds, 8 tens, and 7 ones.

Although this activity is tangential to the typical middle school curriculum, it promotes flexible numerical thinking and even plants the seeds for topics in high school algebra. For instance, polynomials can be interpreted as “base *x*” numbers, where the “digits” are no longer restricted to the numerals 0 through 9. It also highlights that mathematics is a deeply human endeavor, shaped by unique and sometimes surprising choices in language, terminology, and notation.

The content of this chapter serves as a great precursor to the study of functions. After all, a function is simply a leashing pattern from one set (the **domain**) to another set (the **codomain**), but with less strict rules—more than one object in the domain can be leashed to the same object in the codomain, and some elements in the codomain can be skipped.

I find this explanation of infinity not only motivates students but also provides valuable context and meaning for wanting to explore the concept of a function. It makes for a fun and mind-wowing start (Figure 1.28).

**Figure 1.28** • *Domains and Codomains of an Example Function*




## A TASK FOR HIGH SCHOOL GRADES

Have students work through the content of this chapter before starting a unit on functions. Premade PowerPoint slides can be found at https://companion.corwin.com/courses/mathdecoded.