

WHAT YOUR COLLEAGUES ARE SAYING . . .

Blanton and Gardiner have successfully transformed algebra from a high-school gatekeeper into an elementary-grade gateway. This book is an essential roadmap for school leaders who are committed to equity, providing teachers with the practical “thinking practices” necessary to ensure every student develops the mathematical confidence and foundational logic required for long-term academic success.

Rosamaria Murillo

Principal, Ladera Palma Elementary, La Habra City School District
Whittier, CA

The authors do a fantastic job of helping to make algebra relatable and attainable to everyone in this deep dive into early algebraic thinking. They take the fear of the variable and replace it with a hefty portion of reasoning and understanding. Any educator in elementary math would find this text a valuable resource in guiding them to understand the reasons why it is necessary to teach algebraic skills to even our youngest students.

Hope Edlin

Third- and Fourth-Grade Math, Science, and Social Studies Teacher,
Bethel Elementary School, Greenville County School District
Simpsonville, SC

This book will change the way elementary teachers approach algebra. It makes the content feel attainable and approachable, all in a clear, easy read.

Karina Cousins

Teacher, School District of Palm Beach County
Lake Worth, FL

Part of the reason educators have not made more progress in improving math teaching and learning is that we continue to operate from an incomplete understanding of what math is; a definition originating in our own limited math learning experiences as students. This book shows what real mathematical reasoning can look and sound like. It offers easy-to-implement learning experiences to engage students in algebraic thinking, allowing students to learn mathematics with understanding and recognize themselves as innately math capable.

Sue Chapman

Coauthor, *Power Up Your Math Community* and *Eight Habits of Highly Effective Math Students (and the Teachers Who Teach Them)*
Professional Learning Consultant, Teacher Educator
Houston, TX

This book provides a clear research-based approach to integrating algebraic thinking into the core math curriculum. From a district leadership perspective, I've seen this approach build coherence across classrooms while strengthening both teacher practice and student understanding. It is an invaluable resource for educators committed to building a comprehensive approach to early algebraic reasoning.

Leanne Daughtry

K–12 Math and Science Curriculum Director
Smithfield, NC

Early Algebraic Thinking in Action provides teachers with a roadmap for integrating algebra concepts with their mathematics instruction. The authors explain how integrating algebra and arithmetic deepens students' understanding of arithmetic while also providing a foundation for formal algebra. Useful examples of how teachers can modify their curriculum to "raise the algebra temperature" of their lessons appear throughout the book. This resource is a valuable addition to any teacher's collection.

Linda Levi

Director, CGI Math Teacher Learning Center
Madison, WI

Maria Blanton is the nation's leading expert in promoting children's early algebraic thinking. This book provides actionable ideas that can be used in any classroom, along with the rationale behind them, and is based on decades of research on what works by Blanton and others.

Bethany Rittle-Johnson

Professor and Anita S. and Antonio M. Gotto Chair in Child Development
Nashville, TN

Early Algebraic Thinking in Action: A Teacher's Guide for Grades K–5 is a welcome addition to support teachers with fostering and deepening students' algebraic thinking during the elementary years. I particularly appreciated that the authors provided learning goals to address the Algebraic Thinking Big Ideas by grade band. These learning goals are great entry points for teams of teachers to engage in cycles of inquiry together to intentionally and strategically design sequenced tasks and lessons for their students.

Joseph Espinosa

Fifth-Grade Teacher, Los Angeles Unified School District
San Gabriel, CA

Blanton and Gardiner spark our curiosity about young children's algebraic reasoning and give us tools to explore, appreciate, and support that reasoning. This book is full of interesting tasks, clever adjustments to existing tasks, suggestions for conversation starters, tips for appreciating algebraic ideas in children's informal language, and so much more. The authors provide practical advice for both individual teachers and entire schools so that the power of early algebra can be realized.

Victoria Jacobs

Yopp Distinguished Professor of Mathematics Education,
University of North Carolina at Greensboro
Greensboro, NC

This book focuses on the foundational aspects of learning algebra. It dispels the idea that algebra is about manipulating symbols but rather emphasizes the importance of algebraic THINKING which is critical in building fundamental processes that lead to stronger achievement. I highly recommend the use of this book building an effective mathematics program.

Barbara Dougherty

Professor, Mathematics Education
Key West, FL

A valuable resource for anyone involved in teaching mathematics in the elementary school grades. Content is easy to access and the Take a Look questions provide multiple opportunities for readers to make connections between the book's ideas and readers' individual contexts.

Drew Polly

Professor, University of North Carolina at Charlotte
Charlotte, NC

Early Algebraic Thinking in Action

Early Algebraic Thinking in Action

A Teacher's Guide for Grades K–5

Maria Blanton
Angela Murphy Gardiner

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Mathematics

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Visit the companion website at
<https://companion.corwin.com/courses/EarlyAlgebraicThinking>

Note From the Publisher: The authors have provided web content throughout the book that is available to you through QR (quick response) codes. To read a QR code, you must have a smartphone or tablet with a camera. We recommend that you download a QR code reader app that is made specifically for your phone or tablet brand.

PREFACE

Almost everyone has had an algebra class at some point. For many students, this experience did little to spark their interest in mathematics. For some, it left them with a feeling of failure. Maybe you were such a student. But success in algebra is critically important in school mathematics. Most students are required to pass some form of algebra course sequence to graduate from high school. For college-bound students, a strong preparation through at least Algebra II correlates highly with their graduation from college (U.S. Department of Education, 2008). Students who complete Algebra II are more than twice as likely to graduate from college than students with less math preparation (U.S. Department of Education, 2008). For non-college-bound students, technical schools can require successful completion of high school algebra courses for acceptance into apprenticeship programs (EdSource, 2026). Regardless of career path, the problem-solving and critical thinking skills that come from learning algebra are important for all students' success in future careers (EdSource, 2026).

THE “ALGEBRA PROBLEM” (AND WHAT TO DO ABOUT IT)

The problem is that algebra has been an obstacle for too many students. Traditionally, algebra has been treated as a formal set of courses that students take in middle or high school (Algebra I, Algebra II, and so on). But moving from arithmetic in elementary and middle grades to formal algebra courses has often been rocky for students. You might have experienced this yourself. Many students have found themselves suddenly trying to navigate unfamiliar content where mathematical ideas were expressed with letters, quantities were “unknown,” and instruction focused on manipulating symbols through a new and strange syntax. The large failure rates in high school algebra that resulted from this (Stigler et al., 1999) are no surprise. Algebra had become a “gatekeeper” that restricted students' passage into “every avenue of the job market and every avenue of schooling” (Schoenfeld, 1995, p. 11).

As educators began to look for new pathways to improve students' success in algebra, it became clear that we needed to change *how* algebra is taught and *where* it is taught. Now, algebra is no longer seen as just a “high school course.” Instead, it is a strand of learning across school mathematics that begins in kindergarten (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010).

What does this mean? If you are an elementary teacher, you probably never imagined you would be asked to teach algebra! What would algebra in the elementary grades (or, *early algebra*) even look like? Would it amount to “pre-algebra” ideas in middle grades repackaged for young learners? Are elementary students capable of thinking in ways that have traditionally been viewed as possible only for older students? How can elementary teachers be supported in building vibrant classrooms where algebraic thinking thrives?

These are important questions. In the years since these questions were raised, we’ve gotten a much better picture of how they might be answered. For example, we know that early algebra helps students develop a strong relational understanding of the equal sign (Blanton et al., 2019; Carpenter et al., 2003; Stephens et al., 2017). Without this, students can struggle in later grades with algebraic processes such as solving equations and representing functional relationships (Knuth et al., 2006; Strachota et al., 2017). We know that students can learn to use variable notation to represent properties of operations, unknown quantities in expressions, and functional relationships (e.g., Blanton et al., 2019; Carraher & Schliemann, 2007; Schifter, 1999). They can use sophisticated representations like graphs and tables to reason with functional relationships (Stephens et al., 2017). They can reason with arithmetic relationships to compute more efficiently (Blanton et al., 2011; Carpenter et al., 2003). And they can learn to build representation-based mathematical arguments (Schifter, 2009) to show why general relationships are true. This isn’t just important algebraic thinking; it’s good *mathematical* thinking!

HOW DO WE KNOW THIS APPROACH WORKS?

Early algebra’s potential for changing students’ success in mathematics is clear. But are the trade-offs of time, effort, and resources needed to implement early algebra well in elementary grades worth any gains students might make? In other words, does it work?

To look for answers to this question, we developed a Grades K–5 early algebra curriculum that could supplement students’ regular math curriculum (see Blanton et al., 2020a, 2020b, 2020c; Blanton et al., 2022a, 2022b, 2022c). We’ve learned that regular math curricula are often too focused on arithmetic concepts and don’t adequately prepare students for algebra. We needed an approach that could. Through a large-scale randomized study (the “gold standard” of evidence), we found that elementary students who were taught our curriculum as part of their regular math instruction showed significantly better understanding of core algebraic concepts compared to their peers who were taught only their regular curriculum

(Blanton et al., 2019). Even as these students moved through middle grades, the differences in their understanding of algebra remained significant (Stephens et al., 2021).

Think about this. Our research shows that elementary students from diverse backgrounds, with various learning strengths and differences, are significantly better able to reason algebraically when elementary teachers have good, effective early algebra resources to use. Students are also significantly better prepared for algebra in later grades.

Over the years, the extent of elementary students' ability to think algebraically has surprised us. What's more, we've found that they are enthusiastic learners. We're convinced that early algebra helps build confident students who can think mathematically with understanding. Early algebra *does* matter. And we think this is a story worth telling.

WHY WE'VE WRITTEN THIS BOOK

In this book, we share what we have learned from years of classroom research. We hope that the ideas here will help you in two ways: (1) build your understanding of opportunities for algebraic thinking in your own curriculum; and (2) teach you practical strategies for designing early algebra lessons or activities that you can integrate into your regular math instruction.

There are already many good resources available that can help you navigate early algebra. What makes this book unique? First, *our book is comprehensive in its treatment of early algebra content*. Algebraic thinking can happen in different types of mathematical content. It's important not to miss any of them! We've distilled years of research (others' as well as our own) into three main areas that we refer to as "Big Ideas." Chapters 2–4 are organized around these Big Ideas. They will give you a comprehensive approach to early algebra in your classroom.

Our book is comprehensive in its approach to algebra across the elementary grades. Early algebra needs to be coordinated across elementary grades to be effective. You might be very enthusiastic about early algebra. That's great! But without support in the years before and after, your students will have a hard time building real understanding. We give you a "roadmap" of learning goals to guide you in what you might focus on *within* a grade and how to connect these ideas *across* grades. Understanding how to sequence algebraic ideas throughout elementary grades is the starting point to effective change.

Our book is based on an effective approach to improving children's algebraic thinking. We have spent almost two decades in elementary classrooms, observing, analyzing, listening, teaching, and, most importantly, learning. We've talked with teachers,

students, curriculum coordinators, math coaches, and district leaders. We've tested ideas, tasks, lessons, and the boundary points of what we thought children might be able to do. Our research has shown that our approach is effective (Blanton et al., 2019).

WHAT TO EXPECT NEXT

Are you ready to get started? We know that the idea of teaching “algebra” as early as kindergarten can seem overwhelming. What concepts should be taught? At what grades? How should ideas be developed across grades? How can tasks be designed and sequenced to build understanding? How does your regular curriculum teach core early algebraic concepts, if at all? Here's what you can expect in the following chapters to help you answer these questions.

A Different Way to Think About Algebra

In Chapter 1, we lay out a framework for early algebra as a set of four mathematical thinking practices: generalizing, representing, justifying, and reasoning with mathematical structure and relationships. In Chapters 2–4, we explore these practices within our three “Big Ideas”: (1) equivalence, expressions, equations, and inequalities; (2) relationships in arithmetic; and (3) functional relationships. This might be a different way for you to think about algebra. We hope it will be a fresh approach to seeing algebra *through its connections to arithmetic*, not as a collection of procedures for manipulating symbols.

Using tasks, vignettes, and lesson ideas drawn from classroom research, we explore in Chapters 2–4 how early algebra content connects to the arithmetic you already teach. We begin each of these chapters with a vignette that depicts conversations with students about a particular task. The task highlights the key focus of each chapter. We look at how different tools can help young learners make sense of algebraic ideas. We provide research-based insights into students' algebraic thinking. We offer teaching strategies that can enrich your lessons and engage learners.

We also give you opportunities to pause and look for early algebra “gaps” in your current curriculum. We help you think about how to address these. Gaps in the treatment of early algebraic concepts and practices across elementary grades will leave students ill-prepared for algebra in later grades. In a recent study, we found that only about 30% of Grade 3 students understood that the Commutative Property of Addition is true for any two numbers. Only 20% could solve the missing value equation $7 + 3 = ___ + 4$, showing they understood the equal sign as a relational symbol. Virtually no students (2%) could build a good mathematical argument to justify a general claim about the sum of odd numbers (Blanton et al., 2019). Even though

students were taught from widely adopted curricula, they still lacked critical algebraic (and arithmetic!) understandings.

Learning Goals as a Roadmap for Early Algebra

As you will see in this book, our early algebra goals are simple: ***Help students learn to generalize, represent, justify, and reason with mathematical structure and relationships.*** This won't happen in a few lessons or in a single grade. Algebraic thinking thrives best through a whole-school approach that is connected across grades and nourished through professional learning communities. It needs a school-wide strategy for *all* elementary grades. There are many great stand-alone early algebra tasks, but using these without a systematic, multigrade approach to early algebra is likely not effective and won't lead to meaningful growth in students' algebraic thinking.

Where does this leave you? What should you do in your classroom? What should you do in your school? This book helps guide you through this process. In Chapter 2–4, we identify *learning goals* for each Big Idea and *key steps* you can take to build early algebra content around these goals. We describe why each learning goal matters in children's algebraic thinking (*Why This Matters*) and identify a sequence of key steps or actions that you can use to structure lessons around a given goal (*What You Can Do*). Use the learning goals and their goal-related actions to guide you in creating a sequence of lessons around algebraic concepts and practices.

The learning goals are designed to help you develop students' algebraic ways of thinking, not a set of skills or procedures to perform typical algebra tasks. The content and sequencing of these goals is based on learning progressions in how children think algebraically. They can help you re-imagine the role of early algebra in your classroom. They can also help you create a school-wide approach for developing children's algebraic thinking.

Think of the learning goals as a roadmap of ideas for building early algebra lessons. We've arranged them so that concepts build over time. We've also organized them by grade bands (Grades K–2 and Grades 3–5) to provide some structure. You might notice that the “grain size” of learning goals is different for these grade bands. Younger learners need smaller goals. As students mature, the complexity of learning goals will increase.

Waypoints to Pause, Reflect, Learn

We've included different waypoints throughout the book to help you make sense of the ideas here. *Key Idea* summarizes important points to keep in mind. *Just So You Know* provides you with important background around mathematical content that might be less familiar to you.

Some of our waypoints (*Take a Look*, *Think About It*, and *Try It*) give you a chance to put ideas into action. We sometimes revisit similar themes in different waypoints. Just like your students, rethinking ideas along the way is helpful as your understanding about early algebra grows and changes.

Try It gives you simple entry points for testing algebraic ideas in your classroom. Some are very quick to implement. Some will take a little more time as ideas become more complex. Use these to explore “small-sized” algebraic concepts with students before you teach a formal lesson. You might try them as part of a daily math routine. As you do, think about what you learn about yourself and your students. What surprises you?

Take a Look and *Think About It* will give you a deeper “dive” into the ideas in this book. It’s important to take time to do this. *Take a Look* asks you to explore how your curriculum treats different early algebra concepts. Take time to reflect on these. They will help you identify early algebra “gaps” so that you can plan how (and where) to integrate algebraic thinking into the arithmetic you’re already teaching. *Think About It* asks you to solve math tasks, reflect on how arithmetic and early algebra are connected, examine how your practice supports algebraic thinking, or try your hand at designing tasks or lessons. Both *Take a Look* and *Think About It* will give you important knowledge around increasing early algebra’s role in your classroom. These might best be explored through professional learning communities at your school so that you can share this process with other teachers.

As you work through the ideas in this book, take time to engage with each of these waypoints. Give yourself space to pause, learn, and grow. Reflect on your practice and how it changes. Do you focus on asking questions and listening to students’ thinking? We hope that what you learn here inspires a journey in your own thinking that redefines how you view algebra and transforms how you teach it. Your students will reap the rewards!

ACKNOWLEDGMENTS

What you read in this book represents much more than the work of two people. It's the culmination of years of effort by numerous researchers, classroom teachers, school and district leaders, and—of course—students. When we began our search for effective methods to teach algebra in the elementary grades, we had little idea how many years it would take to get to this point. We are deeply grateful for the contributions of our colleagues and collaborators, Dr. Barbara Brizuela, Dr. Eric Knuth, Dr. Ana Stephens, Dr. Rena Stroud, Dr. Susanne Strachota, and Dr. Despina Stylianou, as well as the many graduate and postdoctoral students who have been part of this journey. We owe a significant debt of gratitude to this dedicated team of researchers who have collaborated with us along the way. They share equally with us in the fruition of this work.

Elementary teachers are at the forefront of changing what algebra looks like in school mathematics. We are grateful to the many teachers who have invited us into their classrooms to observe, to learn, and even to teach. They've helped us think through what works and what doesn't. They've inspired us with their willingness to try new approaches to teaching mathematics. Their commitment to creating classrooms that foster inquiry and conversation around children's algebraic thinking has been essential. These classrooms have been the laboratories for the ideas in this book.

We would like to thank the Corwin editorial team, along with anonymous reviewers of earlier drafts of this book, for the insights, advice, and creative guidance they provided. We especially want to thank Debbie Hardin for her editorial expertise and consistent support in guiding this book from an idea into the final product. We are particularly thankful for her commitment to creating resources to help elementary teachers build children's algebraic understanding.

Ultimately, this book is about students. Some might think that algebra in the elementary grades is only for certain students. It's not! We want to be clear on this point. All elementary students can learn to think algebraically. Our work has embraced inclusive classrooms so that every student can be a participant. And you might be surprised: We've found that young learners enjoy algebra! Through the enthusiasm that early algebra inspires, elementary students have shown us what it can look like to engage joyfully in thinking mathematically. We are most thankful for that.

—Maria Blanton and Angela Murphy Gardiner

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Dr. Maria Blanton has conducted research on teaching and learning algebra in the elementary grades for almost three decades. During that time, she has served as a professor of mathematics and STEM education at the University of Massachusetts Dartmouth and, for the last 15 years, as a senior scientist at TERC, Inc. She has led numerous federally funded projects to explore the cognitive foundations of children's algebraic thinking and to design effective methods for developing children's algebra readiness for middle grades. As part of this, she

coauthored *LEAP: Learning Through an Early Algebra Progression*, an early algebra curriculum for Grades K–5. She is also coeditor of the research volumes *Algebra in the Early Grades* and *Teaching and Learning Proof Across the Grades: A K–16 Perspective*. She is author of *Algebra and the Elementary Classroom: Transforming Thinking, Transforming Practice* and coauthor of *Developing Essential Understanding of Algebraic Thinking for Teaching Mathematics in Grades 3–5* and *Teaching with Mathematical Argument: Strategies for Supporting Everyday Instruction*. She is the recipient of the 2020 International Award for Interdisciplinary Excellence in Mathematics Education (Texas A&M University) for her research on children's early algebraic thinking.



Angela Murphy Gardiner is a research and design specialist at TERC, Inc., in Cambridge, Massachusetts. Before becoming a full-time researcher, she was an elementary classroom teacher in Grades 1–3. Her current work bridges research and practice, drawing on her classroom experience to inform the design and study of instructional approaches that support young students' mathematical thinking. She focuses on identifying effective ways to foster early algebraic reasoning and prepare students for later success in mathematics. Angela has collaborated on numerous federally funded research projects exploring the development of children's algebraic thinking

and understanding. She is also coauthor of *LEAP: Learning Through an Early Algebra Progression*, a research-based early algebra curriculum for Grades K–5.

HOW TO USE THIS BOOK

If . . .	How This Book Can Help . . .
You are looking for an effective approach that can help all children be successful in algebra.	There are many good early algebra resources that can help you increase the “algebra temperature” in your classroom. This book offers that and more. The methods you will learn about here are based on a classroom-tested approach that has been shown to be effective in improving all children’s algebra readiness for middle grades.
You want to improve children’s mathematical reasoning, not just their fluency with skills and procedures.	Thinking and reasoning mathematically are essential to what it means to do mathematics. This book treats early algebra as a set of four mathematical thinking practices: generalizing, representing, justifying, and reasoning with mathematical structure and relationships. When children engage in these practices, they are thinking and reasoning mathematically.
Your regular curriculum has gaps in how it develops children’s algebraic thinking and you are looking for resources to address this.	This book gives you a step-by-step guide for how to build early algebra activities and lessons into your regular curriculum. It includes engaging and thoughtful tasks, suggestions for how to use tools to support students, helpful strategies for teaching early algebra, and classroom conversations that show students’ early algebraic thinking.
You want content that deeply aligns with the mathematical ways of thinking identified in NCTM’s <i>Process Standards</i> and Common Core’s <i>Standards for Mathematical Practice</i> .	Early algebra is one of the best ways to address important process and learning standards. As you will see in this book, the connections between these standards and children’s early algebraic thinking are deep.

(Continued)

(Continued)

If . . .	How This Book Can Help . . .
Your school or district wants a comprehensive treatment of early algebra across elementary grades.	Algebraic thinking thrives best through a school-wide approach that is connected across grades and nourished through professional learning communities. This book will help you understand critical learning goals for each grade in elementary school (K–5). It provides you with specific steps you can take to design lessons that address these goals. It is organized around three key Big Ideas for a comprehensive treatment of early algebra content.
You want your students to understand arithmetic more deeply.	For too long, educators viewed school mathematics as “arithmetic first, algebra later.” The truth is that children cannot truly understand arithmetic without understanding the underlying structure that governs operations on numbers. Early algebra does this. This book gives you a fresh way of seeing algebra through its connections to arithmetic. Students who think algebraically about number and operations have a better understanding of arithmetic.

Chapter

1

Introduction

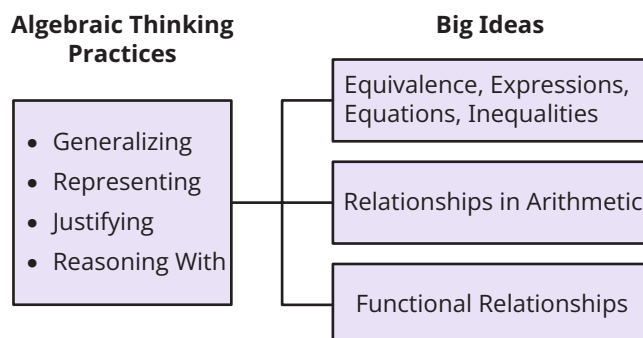
IN THIS CHAPTER YOU WILL . . .

- Reframe your understanding of algebra as a set of mathematical thinking practices rather than a set of mathematical skills and procedures
- Learn about core areas of mathematics (“Big Ideas”) where students can think algebraically

WHAT IS EARLY ALGEBRA?

Introducing algebra as early as kindergarten (e.g., NGA Center & CCSSO, 2010) raises a fair question: “*What algebra?*” What is the algebra we want young children to learn? How would you define algebra? If you think of algebra as largely about manipulating variables to simplify difficult expressions or solve complicated equations, you might rightly question whether young children should spend their time on this in math class. We agree! Algebra in the elementary grades—that is, *early algebra*—is not “algebra early” (Carraher et al., 2008). It isn’t about repackaging all the procedures and processes you might have learned in an algebra class for young children.

NCTM’s *Principles and Standards* (NCTM, 2000) states that students who “reason and think analytically tend to note patterns, structure, or regularities” (p. 56). This captures the intent of early algebra: developing mathematical thinkers who notice, understand, and can reason with mathematical structure and relationships. We frame early algebra as a set of four key algebraic thinking practices: generalizing, representing, justifying, and reasoning with mathematical structure and relationships (Blanton, Brizuela et al., 2018). In the elementary grades, these practices occur primarily in several core content areas that we call “Big Ideas” (see Figure 1.1). In this chapter, we briefly introduce the four algebraic thinking practices and describe the Big Ideas. We will look at these more closely in Chapters 2–4.

Figure 1.1 • Algebraic thinking practices and the Big Ideas where they can occur

Generalizing Relationships

The practice of *generalizing* is so important that it has been described as the heart of algebraic thinking (Cooper & Warren, 2011). When students generalize, they notice a general mathematical relationship that *might be true across some domain or set of numbers*. In other words, they see the underlying relationship that connects a set of specific cases in a particular way. For example, they don't simply see that $5 + 4$ is odd. They notice that the sum of *any* odd number and *any* even number is *always* odd. They don't just see that 3×12 is the same as 12×3 because the order of the factors is switched. They recognize that the order in which they multiply *any* two numbers does not change their product. They might notice that the number of bicycle wheels is always double the number of bicycles, no matter the number of bicycles.

Generalizations are the building blocks of thinking algebraically. When students notice general relationships, they are looking for and making use of structure, an important *Common Core* mathematical practice (NGA Center & CCSSO, 2010). When their generalizations arise from "real world" contexts, like functional relationships, they are engaging in another important mathematical practice—modeling with mathematics (NGA Center & CCSSO, 2010). Students of all ages should be given opportunities to generalize.

TIPS

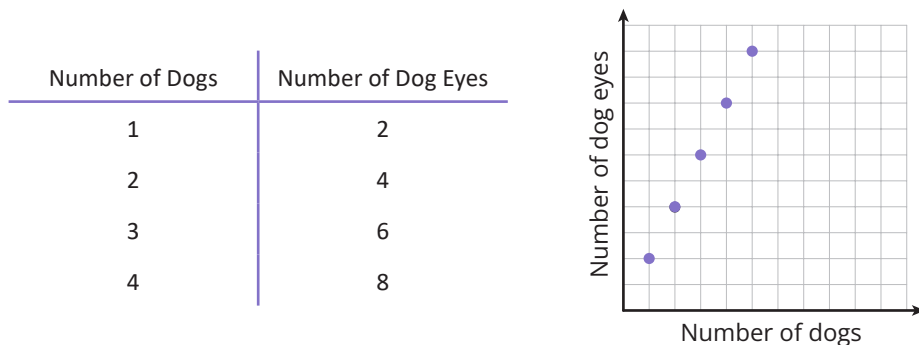
KEY IDEA: *Generalizing* is the heart of algebraic thinking. Students are generalizing when they notice mathematical structure or general relationships.

Representing Relationships

Once students notice a relationship, they can *make a conjecture* to describe what they see. The *Principles and Standards* (NCTM, 2000) is clear: Students should “make and investigate mathematical conjectures” (p. 57). Conjecturing is a way to introduce a general claim, or *generalization*, into classroom conversations. Students usually do this by stating a conjecture in their own words. Using natural language allows them to express generalizations in ways that make sense to them. For example, as a child explores subtraction, they might observe, “I noticed that any time I take away a number from itself, I get zero.”

Students can represent relationships in a variety of ways. In addition to their own words, students might use tables, graphs, pictures, or even variable notation. They might represent the claim, “If I subtract a number from itself, I get zero” with variable notation as $a - a = 0$, for any number a . They might describe the relationship between the number of dogs and number of dog eyes in words as “The number of dog eyes is double the number of dogs.” They might represent this relationship using variable notation as $n = d + d$ or $n = 2 \times d$, where d represents the number of dogs and n the number of dog eyes. They might also represent this relationship in a table or with a coordinate graph (see Figure 1.2).

Figure 1.2 • A table and graph representing the “dog eyes” relationship



Students who can represent relationships in different ways and who understand how these representations are connected have a richer understanding of the relationships (Brizuela & Earnest, 2008). For example, they understand that the way the number of dog eyes increases in the table will result in the graph of a line that shows “steady” growth (see Figure 1.2). Encouraging students to use different representations in this way and to think about how these representations relate to each other is important for developing *representational fluency*.

TIPS

KEY IDEA: Students can represent mathematical relationships in different ways using words, variable notation, tables, pictures, and graphs. *Representational fluency* means understanding different ways to represent mathematical relationships and how these representations are connected.

Variable notation is the representation that might seem the most out of place in elementary grades. Many people think using letters to represent mathematical quantities is only for older students. Given the challenges older students are known to have with variable notation, wouldn't younger children have even more difficulties? After almost two decades of research on variable notation in elementary classrooms, we have found the opposite to be true (e.g., Blanton et al., 2017; Blanton, Stephens et al., 2015)! *As elementary students explore and use variable notation, they develop meaning for it and can use it to represent mathematical relationships.* You might be surprised to learn that elementary students sometimes use variable notation more easily than their own words (Blanton et al., 2019)!

If this seems counterintuitive, think about it this way: How would children make sense of highly complex language in written form if they were not introduced to the “squiggles” that they come to know as letters at an early age? For example, a child has no prior knowledge of the marks that form the letter A. Nor is there anything inherent in the shape of the marks that shows what “A” means. Yet parents start introducing letters to their children long before formal schooling begins. Why? Because as children interact with letters in context (for example, seeing the letter “A” associated with a picture of an apple and hearing the short A sound the letter makes), they come to develop meaning for this symbol. Imagine how children would fare in their understanding of written language if they were not introduced to letters until middle grades! The challenge of developing meaning for letters, understanding how to combine letters to form words, and finally, combining words to form sentences using an (arbitrary) system of rules, would be huge. In the same way, variable notation is an important tool for students to learn to use in elementary grades, long before formal algebra.

TIPS

TAKE A LOOK: What kinds of representations does your curriculum ask students to use? What are students being asked to represent? Are students asked to use words, variable notation, tables, drawing, or graphs to represent general claims?

Justifying Relationships

Generalizing and representing relationships are not the only ways children can think algebraically. When students make a conjecture about a mathematical relationship, they are making a claim about what they think *might be true*. In this way, a conjecture acts as a “major pathway to discovery” (NCTM, 2000, p. 57). Knowing whether a conjectured relationship is true is just as important as making the claim.

TIPS

KEY IDEA: When students make a conjecture about a mathematical relationship, they are making a claim about what they think *might be true*.

TIPS

THINK ABOUT IT: Students sometimes make general claims that aren’t true. For example, they sometimes say that multiplying any two numbers always makes a “bigger” number. Why is this claim false? What is a generalization that your students have made that is not true? How did you address this with your students?

Building viable arguments to show a conjecture is true is an important mathematical practice (NGA Center & CCSSO, 2010). From the early elementary grades, students should experience mathematics as a sense-making activity in which understanding *why* things work is central to *doing* mathematics. Without this as part of their early learning experiences, students will likely have more difficulties with mathematical content in later grades (Knuth et al., 2002; Stylianides & Stylianides, 2008).

Some arguments are meant to justify claims made about specific cases. For example, students might be asked to explain why $3 + 4$ is an odd number. They might argue that 7 is an odd number because 7 cubes cannot be separated into pairs of cubes or into two equal groups:



These arguments are important to make, but they are different than an argument that shows a *general* claim is true. Arguments showing a general relationship is true must show it's true for *all* cases on which the claim is made. But many of the relationships that students notice are claims made for an infinite set of numbers. For example, the conjecture “any number subtracted from itself is zero” is a claim made for all real numbers. How is it possible to show a conjecture is true for every single case when there are an infinite number of cases? What makes a *good* argument for why a mathematical relationship is *always* true?

Students often use *empirical arguments* to show a claim is true. Empirical arguments are based on testing whether some (not all) examples of a conjectured relationship are true or false. For example, students might test several cases of the conjecture “the sum of any whole number plus 2 is even.” Suppose they test this claim by adding 2 to the whole numbers 4, 8, and 10: $4 + 2$, $8 + 2$, and $10 + 2$. Since the sum of each of these is even, they might decide that the conjecture is true. Students find this type of argument very convincing. But is an empirical argument a good one? In this illustration, students might have randomly picked 4, 8, and 10, not realizing that their choice of numbers led them to think the conjecture is true when it is not. If they had chosen an odd number, such as 7, they would see that the conjecture is not true since $7 + 2$ is not even.

A mathematical argument can be convincing to students but still not a very good one. To be clear, looking at different cases or examples of a mathematical claim can help students make sense of the claim (Knuth et al., 2019). For example, if students don't know the answer to the question “Is the sum of two odd numbers even or odd?,” then looking at different cases of adding two odd numbers would be a good way to figure this out. They might then make a conjecture, such as “If two odd numbers are added, their sum is always even.” But an argument that only tests some cases of a claim is not a good mathematical argument because it doesn't show the relationship is true *for all cases*. (Keep in mind that a good mathematical argument is different from arguments students make in science class! In science, arguments are often empirical. Hypotheses are tested through observations, and conclusions are made by considering “cases.”)

TIPS

KEY IDEA: A good mathematical argument must show that a conjecture is true *for all* values for which the claim is made. Arguments that just look at some specific cases of a claim might be convincing, but they are not good mathematical arguments.

What options do young learners have for building good arguments? They don't yet know the methods older students might use to build formal arguments, yet testing examples is not the basis of a good argument. It turns out, they already have tools on hand and ways of reasoning they can use to build good arguments! As early as kindergarten, children can start to build *representation-based arguments* (Schifter, 2009) to show relationships are true. As we explore later in this book, representation-based arguments do not rely on testing examples. They use diagrams, manipulatives, and story contexts—tools accessible to young children—to show why a conjecture is true *for all cases* on which it is made. Building good arguments for why a general claim is true is important mathematical thinking.

TIPS

TAKE A LOOK: What kinds of mathematical claims does your curriculum ask students to make? Does it ask students to make general claims for a class of numbers? Does it ask students to build arguments to support their claims? If so, what kinds of arguments?

Reasoning With Relationships

Reasoning with general relationships is another important way to think algebraically. Ultimately, we want students to be able to use the relationships they notice to solve new math problems. Using structure in this way is an important mathematical practice (NGA Center & CCSSO, 2010). For example, if students know that the sum of two even numbers is even, they might use this to think about whether the sum of three even numbers is even or odd. Another way to reason with relationships is to use properties of operations—even intuitively—to make computational work more efficient (Blanton et al., 2011). For example, a student might compute $4 + 8 - 4 - 8$ by first finding $4 - 4$ and $8 - 8$. Even if the child can't explicitly describe their actions, they are reasoning with properties such as the Commutative Property of Addition and the Additive Inverse Property. This shows a much deeper understanding of arithmetic than simply computing left to right.

TIPS

KEY IDEA: Children are thinking algebraically when they reason with established relationships, such as arithmetic or functional relationships, in new problem situations or in computational work.

NCTM's *Principles and Standards* (NCTM, 2000) states that making sense of mathematics is a process of “developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas” (p. 56). Certainly, this includes generalizing, representing, justifying, and reasoning with mathematical structure and relationships! These practices are essential for algebraic (*and* arithmetic) thinking. And they help students make sense of mathematics. We will explore these practices throughout the remainder of this book, so keep their meanings in mind.

WHERE ARE OPPORTUNITIES FOR ALGEBRAIC THINKING PRACTICES IN YOUR CURRICULUM?

We've described early algebra in terms of four algebraic thinking practices. You might be surprised at the many places where they can occur in your instruction. You've probably already thought of a few! Let's think about this question: What type of mathematical content provides a productive space for children to think algebraically? In our research (e.g., Blanton, Brizuela et al., 2018), we focus on three *Big Ideas* where algebraic thinking can occur (see Figure 1.1). There are practical reasons beyond the scope of this book for why we focus on these three Big Ideas. Keep in mind, though, that algebraic thinking practices can occur wherever generalizations happen. This includes important areas such as geometry and measurement not addressed in this book. In Chapters 2–4, we look at how the four algebraic thinking practices operate within each of our three Big Ideas. For now, let's think about what each Big Idea means.

Equivalence, Expressions, Equations, and Inequalities (Chapter 2) focuses on understanding the relational meaning of the equal sign. This is a foundational concept in both arithmetic and algebra. This Big Idea also addresses mathematical expressions and relationships between quantities (equations and inequalities).¹ Expressions, equations, and the equal sign are building blocks for representing mathematical relationships in arithmetic *and* algebra. It's hard to imagine doing mathematics without them!

Relationships in Arithmetic (Chapter 3) refers to understanding and reasoning with structure in operations on numbers. This involves properties of operations, such as the Commutative Property of Addition, that govern how numbers behave and relate to each other. It also includes less formal types of relationships in operations on numbers, such as “multiplying any two numbers greater than one always results in a larger number.” It involves relationships on classes of numbers, such as even and odd numbers. For

¹We focus primarily on equations, expressions, and the equal sign in Chapter 2. Inequalities are discussed as they relate to these concepts.

example, the relationship “the sum of any two consecutive numbers is always odd” is an important relationship about even and odd numbers.

Functional Relationships (Chapter 4) addresses an important route into early algebra (Carraher & Schliemann, 2007). Chapter 4 examines relationships between two quantities that vary in relation to each other. For example, the number of dog eyes varies in relation to the number of dogs. The number of bicycles wheels varies in relation to the number of bicycles. This Big Idea also includes understanding how to use functional relationships to explore and predict function behavior.

TIPS

THINK ABOUT IT: Where do you see opportunities in your curriculum for engaging students in the algebraic thinking practices identified here?

Reflecting on This Chapter

We’ve found that reframing early algebra as a set of four thinking practices—generalizing, representing, justifying, and reasoning with mathematical structure and relationships—is a powerful lens for early algebra. There are several points about this to keep in mind as you read this book. First, you will likely notice that you are already doing “algebra” with your students. The goal of this book is to help you see this and find ways to extend it. Second, the ideas in this book have been designed through years of classroom research with students across all demographics and with different learning strengths. We know that *all* students can learn to think algebraically. Early algebra is not just for some of your students. Finally, early algebra is not an “add-on” to your curriculum. It builds children’s understanding of algebra *and* arithmetic at the same time. We think you will find that building classrooms where early algebraic thinking thrives is time well spent!

