1 What Is Visible Thinking?

Visible thinking, the focus of this book, may be described as clarity and transparency in one's cognitive processes. Visible thinking requires overt, conscious, and deliberate acts by both students and teachers. When thinking is visible, participants are aware of their own thoughts and thought processes, as well as those of the individuals with whom they are working. With visible thinking, there is a heightened level of awareness both individually and collectively. There is also a heightened degree of productivity referred to as synergy. Visible thinking occurs routinely in effective business communities during dialogues and discussions, brainstorming sessions, collaborative group situations, and crisis-management scenarios. Effective communication is the basis for effective visible thinking. Ideas are formulated, expanded, and refined through sharing. Acquiring this vital skill should not be left to chance.

True mathematical learning, as identified in numerous reports by the National Council of Teachers of Mathematics (NCTM; 2000) and the National Research Council (NRC; 2000, 2001, 2005), requires visible thinking. Research shows that, in the mathematics classroom, visible thinking is the key to mathematics learning and success. Evidence of visible thinking is apparent during mathematical discussions, explanations, demonstrations, drawing, writing, and other ways that ideas are conveyed.

Students and teachers must think, have awareness of their thinking, organize and clarify their thinking, and then share their thinking. Visible thinking is intentional and manifests itself within classrooms in multiple ways:

- Teachers explain their thinking out loud.
- Students orally articulate their thinking.
- Students listen to other students articulate their thinking.
- Students engage in discussions while forming their understanding.

- Students consciously activate their inner dialogue
  - when reading for understanding and
  - when studying mathematics.
- Students record their thinking by
  - solving problems,
  - keeping journals, and
  - completing projects.
- Students demonstrate their thinking through use of technology, manipulatives, or mathematical tools.

Visible thinking occurs within group settings as well as in individual settings. Experts in a field of study are very aware of their knowledge and are very adept at comparing their knowledge with the needs of a situation or problem. "In research with experts who were asked to verbalize their thinking as they worked, it was revealed that they monitored their own understanding carefully, making note of when additional information was required for understanding, whether new information was consistent with what they already knew, and what analogies could be drawn that would advance their understanding" (NRC, 2004, p. 18). These skills and self-monitoring processes used by experts are the very same ones students need to learn and understand mathematics.

When visible thinking is present in classrooms, students are consciously aware of their current understanding of the mathematical concepts being discussed. They are also aware of these concepts in relation to their previous learning and understanding. When thinking is visible, discrepancies and dissonance are obvious to the students. If classroom conditions support visible thinking through safe, open discussion and discourse, these misunderstandings are also readily apparent to teachers. Immediacy is a very important factor in visible thinking. When the discrepancies are apparent to teachers, the teachers have the information they need to take action—and they can clarify the misunderstandings on the spot.

Yet thinking is all too often invisible in schools, and successful learning depends on reversing this trend (Perkins, 2003). "Fostering thinking requires making thinking visible" (Ritchhart & Perkins, 2008, p. 58). By increasing thinking, motivation to learn is also increased. Visible thinking improves the ability to learn, and the increased ease of mastering a skill, in turn, provides motivation to continue learning.

UNDERSTANDING MATHEMATICAL CONCEPTS

The problem $3 + 4 = \square$ is not a challenge for adults and is certainly not difficult for the educators reading this book. Nonetheless, this problem is a significant challenge for very young children. The problem requires translating symbols (3, 4, +, =, and □) into number concepts (a quantity of three combined with a quantity of four), combining the number sets (seven objects), and translating the newly formed set back into the appropriate numerical symbol (7). Students need to recognize the mathematical symbols, understand the symbolic relationships, perform the requested procedure, and accurately select the appropriate symbol—all abstract concepts.

The concepts within this problem are profound and serve as a foundation for mathematical learning. The process—using symbols to represent and solve problems—is mathematics. However, establishing this foundation solely upon rote recall—when I see the symbol *3*, and the symbol *4*, with the symbol *+*, I write down the symbol *7*—is like building a house of cards on a ship at sea. All too frequently, a significant wave or swell brings down the house of cards. This wave, referred to in mathematical circles as the "mathematics wall," may be operations with basic facts in third grade, operations with rational numbers in fifth grade, algebraic symbols in eighth grade, or any of the thousands of interrelated mathematical concepts, skills, and procedures. One thing is certainly known. Far too many students hit the mathematics wall at a very young age, most likely around third grade (Boaler, 2008). Obviously, if mathematics achievement is to improve across all cultures and grade levels, this wall cannot remain standing.

THINKING AS A MATHEMATICAL PREMISE

Mathematics educators have come to recognize that the key to removing the mathematics wall is found in the following premise:

> Thinking is a requirement for learning mathematics.

The question derived from this premise, *Is thinking a requirement for learning mathematics?* leads to additional questions:

- What is mathematical thinking?
- Who needs to do the thinking?
- Can mathematical thinking be taught?
- Does all of mathematics require thinking?
- Is thinking about mathematics natural or manufactured?
- Is there one correct thinking process, or are there multitudes of processes?

These are but a few of the questions that arise when teachers and leaders reflect on mathematical thinking. One thing is very clear. This premise, when understood and taken to heart by teachers, can improve teaching methods and, subsequently, have a career-changing impact. When thinking is recognized and accepted as an essential component of learning mathematics, classrooms must change. If thinking is not intentionally planned to occur, then thinking most likely does not occur for a majority of the mathematics students. Perhaps additional information about student thinking is needed to reinforce this premise.

In the NCTM (2000) *Principles and Standards*, we read, "Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking" (p. 52). Furthermore, "mathematical thinking and reasoning skills, including making conjectures and developing sound deductive arguments, are important because they serve as a basis for developing new insights and promoting further study" (p. 15).

Beginning in Grade 2, students are often asked to work problems such as the one provided in Example 1.1. Insight into visible thinking is gained from reviewing and reflecting on typical responses to such a problem and on alternatives to the problem.

Example 1.1 Coin Problem

I have 3 coins, a nickel, a quarter, and a dime. How much money do I have?

A. 15¢ *B. 30¢* *C. 40¢* *D. 45¢*

The answer is 40¢, and the discussion is over.

There is nothing wrong with this problem if it is used to assess acquisition of knowledge at the requested level. The problem falls far short if used to introduce and promote original or early learning about money concepts and computation with money. There is no time or inclination to think about the mathematics. The focus is on operational procedures for an answer. To address these issues, Example 1.2 provides an alternative.

Example 1.2 Alternative Coin Problem

I have 5 coins in my pocket. The coins may only be pennies, nickels, dimes, or quarters. I reach into my pocket and pull out 3 coins. How much money might I have in my hand?

- *What are some different ways I could have 5 coins in my pocket?*
- *With 3 coins, what is the smallest amount of money I might have in my hand?*
- *With 3 coins, what is the largest amount of money I might have in my hand?*

Multiple ideas, discussion, justification, thinking, reasoning, and problem interpretation are the important points. There are numerous correct answers, and minimum incorrect ones. For instance, one student may answer that the smallest amount of money is 3¢ (three pennies), while another student may respond with 16¢ (a penny, a nickel, and a dime). Often, simple changes in the wording of the problems presented or the questions asked provide opportunities for making student thinking visible in mathematics classrooms. The alternative problem becomes a rich one, with multiple entry points for students with a variety of mathematical backgrounds. We will explore the use of this problem further in Chapter 2.

There is tremendous support for an answer of yes to the premise question, *Is thinking a requirement for learning mathematics?* The *Common Core State Standards* (2010) identify practices for students' proficiently learning mathematics. These practices include such elements as making sense, perseverance, abstract quantitative reasoning, constructing arguments, critiquing thinking, and looking for and using patterns. Visible thinking enhances these practices.

Furthermore, the NCTM (2000) *Principles and Standards* states, "The first five Standards describe mathematical content goals in the areas of number and operation, algebra, geometry, measurement, and data analysis and probability. The next five Standards address the processes of problem solving, reasoning and proof, connections, communication, and representation" (p. 7). By identifying and clarifying these process standards, NCTM has taken a clear stand on the position of thinking in mathematics.

Clearly, half of the standards are identified as process ones. These processes encourage students to actively engage in thinking while learning the content contained in the other half of the standards. These standards address the processes—communicating, reasoning, making connections, problem solving, and creating representations—that make mathematics interesting, engaging, and exciting for students. As noted by the NCTM (2009, p. 3) in its position statement *Focus on High School Mathematics: Reasoning and Sense Making*, they are all visible forms of the act of making sense of mathematics.

We want to take a closer look at the NCTM process standards in relation to visible thinking. Effective *communication* is important to thinking and learning. Students need to be able to clearly and precisely explain their thoughts to other students and to their teachers. Also important is the students' ability to conduct effective internal dialogues. This metacognitive ability, the process of thinking about thinking, is important. Metacognition is internal and external. Because it is often internal for many teachers, students may not be aware of how important the process is in learning without direct teacher intervention (NRC, 2000). As Van de Walle (2004) explains, "Metacognition refers to conscious monitoring (being aware of how and why you are doing something) and regulation (choosing to do something or deciding to make changes) of your own thought process" (p. 54). Standing back and observing one's own thinking process is an important skill for learners (Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003). Being aware of one's thinking promotes *reasoning* and forms more solid *connections* between and among mathematics skills and concepts.

Reasoning and making connections are key in learning mathematics. Many of the ideas that are expressed in the NCTM document about reasoning and sense making go hand in hand with ideas we relate about visible thinking: exploring, conjecturing, explaining, and connecting mathematics to existing knowledge.

Problem solving in mathematics generates many positive attributes for students. Students learn to persist because they have more than one way to analyze and solve problems. They gain confidence through being successful. They are able to transfer knowledge into new and novel situations (NCTM, 2000). Through problem solving, students gain facility in translating mathematical *representations* into real-world situations.

VISIBLE THINKING IN CLASSROOMS

We have absolutely no doubt that thinking is required for learning mathematics. The acquisition of mathematical knowledge is vastly different from the acquisition of language. While students do informally acquire some mathematical concepts, such as ideas of shapes, numbers, and measurement, mathematical knowledge, as a whole, is received through formal instruction. Successful acquisition of mathematical knowledge, usable concepts and skills, requires sustained thinking over time. The NCTM (2009) suggests that students need to develop reasoning habits or ways of thinking that become commonplace in inquiry and sense making.

If this is true, then formal education processes must employ strategies and techniques that make student thinking visible to both students and teachers. In other words, in effective classrooms, students' thinking is made visible and feedback is provided. "Given the goal of learning with understanding, assessments and feedback must focus on understanding and not only memory for procedures or facts" (NRC, 2000, p. 128). Failure of instructors to understand student thinking, connections, and conceptual understandings results in learning disasters. An example appropriate for Grade 7 is provided in Example 1.3.

Example 1.3 Proportion Problem

Jill walks 1 mile in 12 minutes, and Jane walks 1 mile in 10 minutes. Jill lives 1 mile from school, and Jane lives 1.5 miles from school. If the girls start home from school at the same time, then who arrives home first?

A. Jill *B. Jane* *C. Tie* *D. Not enough information provided*

The problem is intended to be solved by setting up proportions. Jill lives 1 mile from school and walks 1 mile in 12 minutes, so Jill arrives home in 12 minutes. How fast does Jane arrive home? The proportion is

1 mile is to 10 minutes as 1.5 miles is to x minutes

1 mile/10 minutes = 1.5 miles/x minutes

Students solve by cross multiplying and, if they do it correctly, obtain $x = (10 \times 1.5) = 15$ minutes. Jane arrives home in 15 minutes, and Jill arrives home in 12 minutes. So the answer to the problem is Jill.

What if students realized that Jane walks half a mile every 5 minutes, and therefore walks one and one-half miles in 15 minutes? They have correctly solved the problem but are most likely not aware of the mathematics involved in proportions. In order to better understand proportions, students need more time to think and reason. Therefore, they need to remain engaged in the problem.

Students working in pairs on a problem such as Example 1.4 have multiple opportunities to think about and discuss proportional relationships.

Example 1.4 Alternative Proportion Problem

Jill walks 1 mile in 12 minutes, and Jane walks 1 mile in 10 minutes. Both girls live at least 1 mile from school but less than 5 miles from school.

- *If Jill arrives home first, what distance might the two houses be from school?*
- *If Jane arrives home first, what distance might the two houses be from school?*
- *If Jane and Jill arrive at their homes at the same time, what is the closest the two houses can be from school?*
- *If Jane and Jill arrive at their homes at the same time, what is the farthest the two houses can be from school?*

VISIBLE THINKING SCENARIO 1: AREA AND PERIMETER

Continuing through the chapters in this book, you will see that we have provided a variety of visible thinking scenarios for different grade levels at the end of the chapters. The intent of these student-teacher dialogues is to show how visible thinking might manifest itself in mathematics classrooms. A manifestation highlighted in these scenarios is how teachers can use visible thinking to effectively, quickly, and appropriately intervene with student mathematical misunderstandings.

This scenario involves perimeter and area. In many states, students initially encounter the idea of perimeter in Grades 3 or 4 and continue with various extensions into the middle school. Area concepts typically begin in Grades 4 or 5 and also extend into the middle school. In the NCTM *Curriculum Focal Points* (2006), the study of perimeter as a measurable attribute is suggested as a Measurement Connection to the Grade 3 Focal Points, whereas area is listed as a Focal Point for Content Emphasis in Grade 4. Within the *Common Core State Standards* (2010), perimeter is introduced in Grade 3. The concept of perimeter is combined with area in Grade 4.

Even with these early encounters with both ideas, students still lack an understanding of the difference between perimeter and area.

Problem

A rectangle has a perimeter of 64 inches. What are possible areas for this rectangle?

Mathematics Within the Problem

The teacher is helping students understand area, perimeter, and their relationship. She assigns student pairs to work on the preceding problem. The teacher expects students to find areas randomly at first but then become more organized in their approach. As the students organize their thinking, the teacher will investigate and discuss some patterns with her class. She expects students to recall and understand that the perimeter of a rectangle with length l and width w is $P = 2l + 2w$. In the case where the perimeter is 64 inches, students would establish that $2l + 2w = 64$ inches. This is the same as $2(l + w) = 64$, or $l + w = 32$. If only whole number lengths and widths are considered, then students can set up a table such as Figure 1.1.

Figure 1.1 Area of a Rectangle With Perimeter of 64 Inches

Length (l)	Width (w)	Perimeter (P)	Area (A)
31	1	64	31 in.^{2}
30	2	64	60 in.^{2}
29	3	64	87 in.^{2}
.	.	.	.
.	.	.	.
.	.	.	.
16	16	64	256 in.^{2}
15	17	64	255 in.^{2}
.	.	.	.
.	.	.	.
.	.	.	.
2	30	64	60 in.^{2}
1	31	64	31 in.^{2}

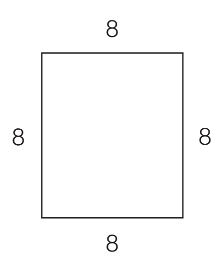
The teacher wants students to understand a significant fact relating perimeter and area for rectangles: For a fixed perimeter, the rectangle with the greatest area is a square. For our particular problem, the greatest area is 16 × 16 = 256 sq. in. The teacher is moving about the room listening to students talk and observing their work.

What Are Students Doing Incorrectly?

The teacher notices a student pair has drawn a rectangle and written an explanation, as shown in Figure 1.2.

Figure 1.2 Students' Reasoning Error

8 × 8 = 64, so the rectangle must have a length of 8 and a width of 8. Therefore, the dimensions must be



The area is 8 + 8 + 8 + 8 = 32 inches.

What Are Students Thinking and Saying Incorrectly?

The teacher asks the students to explain their thinking in solving the problem. The students share their ideas.

> We know that 8 × 8 = 64, so this must be the basis for solving the problem. Since the length is 8 and the width is 8, then the area must be 8 + 8 + 8 + 8 = 32 inches.

The students are distracted by information they know to be true. They know that 8 × 8 = 64. Since the perimeter is 64 inches, students have allowed negative transfer to occur. Because they know this fact, they assume it must play an important role in solving the problem. The students are so convinced of this that they let it overshadow other information they also know.

Teacher Intervention

The teacher bends down to eye level and asks the students to look at her. "Without looking or thinking about this problem, I want you (first student) to explain perimeter and you (second student) to explain area."

The first student responds, "Perimeter is the distance around the outside." The second student responds, "Area is the space inside."

The teacher asks the students to turn their paper over and draw pictures that would show the perimeter of a rectangle and the area of a rectangle. The students draw two rectangles and demonstrate perimeter is the distance around and area is the space inside. The teacher asks, "What measurement units are used for perimeter and what are used for area?" Students correctly identify inches and square inches.

The teacher responds with another question that brings visible thinking to the forefront. "If I make the width of your drawing 2 inches, and the length of your drawing 6 inches, what is the perimeter?"

The first student draws a rectangle, labels the dimensions, and answers, "2 plus 6 is 8, plus 2 is 10, plus 6 is 16. The perimeter is 16 inches."

"What about the area?" asks the teacher. Pointing to the rectangle just drawn, the second student gives an answer: "2 times 6 is 12, so the area is 12 square inches." At this point, the first student sees their error and exclaims, "Oh no, I see what we did! For perimeter, we need to find 2 of the width and 2 of the length that add up to 64 inches. So on this rectangle, if the width is 1, then we have a 1 here and a 1 here (indicating the two widths). So that is 64 – 2, or 62. Half of 62 is 31, so we have 31 here and 31 here (indicating the length). Our perimeter is always 64 inches, and in this case our area is 1 × 31 = 31 square inches for the area."

The teacher asks the second student, "Do you understand, too?" The student replies, "I think so." The teacher encourages the pair to work out a few more examples. "Raise your hand for me to check back with you. Both of you need to understand the problem and the solution. I think you have it. That was good thinking!"

How did the teacher use visible thinking to intervene and correct a misunderstanding?

Students were engaged in a discussion not only between themselves but also with the teacher. They articulated their thinking to the teacher and, as they did so, the teacher was able to diagnose the error in thinking. With students drawing a rectangle and labeling its dimensions, the teacher was also able to understand their thinking and assist them in clarifying the relationship between perimeter and area. She was able to make students aware of their own thinking.

SUMMARY

In this chapter, we have responded to the question "What is visible thinking?" with the answer "a conscious, deliberate set of actions that provides clear evidence of the current level of student knowledge and understanding." The examples that have been provided shed light on what currently happens in much of our mathematics teaching and how opportunities for mathematical learning can be provided for students when adjustments in our teaching practices are made. Student thinking becomes visible when teaching practices

- Make problem solving and use of problem solving strategies a regular focus of student learning.
- Make students aware of their own thoughts and thought processes.
- Make sharing of mathematical ideas an integral part of lessons.
- Make communication both verbal and written.
- Make student thinking visible in classroom discussions of all kinds.

As we continue in the following pages, many of the ideas that have been suggested in this chapter will be expanded. The purposes and positive effects of visible thinking are identified and explained, as are research-based teacher practices that make student thinking visible. Figure 1.3 offers an overview of the benefits for students of visible thinking.

Figure 1.3 Visible Thinking: Purposes and Effects for Students

Visible thinking increases equity, the opportunity for every student to learn mathematics, by

- Increasing student interest, engagement, and motivation
- Promoting connections to previous learning
- Providing opportunities to think deeply
- Encouraging reasoning and sense making
- Opening dialogue and discourse within the classroom
- Promoting conceptual learning
- Increasing student feedback through ongoing formative assessment
- Supporting belief in effort over innate ability
- Broadening student understanding about learning mathematics
- Promoting student responsibility for learning
- Fostering a community of learners