Chapter I

What is Mathematical Investigation?

Problem posing

Conjecture

Habits of mind

Proof

Problem solving

What if ... ?
What if not?

$$1+2+3=6$$
$$10+11+12+13+14+15+16=91$$
$$3+4+5+6+7+8+9+10+11+12=75$$
$$8+9+10+11+12+\cdots+107=5750$$
$$125+126+127+128+\cdots+2003=1999256$$
$$n+(n+1)+(n+2)+\cdots+(n+k)=137$$

Introduction

Current mathematics curricula ask students, from time to time, to "investigate." But how does one do that?

Another question: why all the fuss about investigation? Investigation is not the only way to learn mathematics, nor even the best way in *every* situation.

The ability to investigate a situation is, in itself, an important skill for students to acquire. In mathematics—as in science, or diagnosing the ills of an automobile, a computer, or a person—proper investigation is often the first step in successful problem solving. Furthermore, investigation helps to bring to the fore an essential feature of the subject itself.

Mathematics is a specialization of many of the most powerful thinking techniques people normally use. Part of its great power derives from the facts, formulas, and techniques it provides to the sciences. What makes it of value even to those who will someday forget the facts and formulas is that it highlights, extends, and refines the kinds of thinking that people do in *all* fields. These include investigation, pattern-seeking, and proof.

Proof outside of mathematics is different, in ways, from proof within the discipline, but the fact that the same word is used attests to the relatedness of the many purposes of proof, and even to similarities in the ways of thinking.

Skilled investigators in any field have strategies that go beyond poking around and hoping for the best. In investigation, as in other aspects of thinking, mathematics adds its own special features. What makes an investigation *mathematical*? What's next after finding a great pattern?

In *What is Mathematical Investigation?*, you will take a mathematical investigation from start to finish—from exploratory stages through reporting logically connected results—and you will find strategies that you can use with your students to develop their investigative skills.

You might also encounter some new mathematical facts and relationships, but the real purpose is for you to investigate *investigation*.

1. Problem solving and problem posing

Take 10 to 15 minutes for a preliminary exploration of the problem below—just long enough to develop some initial conjectures.

Such a tiny amount of time is not nearly adequate for a thorough look at this investigatory problem, but even 15 minutes should give you a sense of what students begin to see as they explore it. For the moment, this glimpse is enough.

Students in the first year course of the Interactive Mathematics Program (IMP) *are given three days to explore this lovely problem. You will get the chance to explore this problem in greater depth later.*

In the brief time you devote to the problem now, keep track of partial answers and any new questions that may come up.

PROBLEM

1. The number 13 can be expressed as a sum of two consecutive counting numbers, $6+7$. Fourteen can be expressed as $2+3+4+5$, also a sum of consecutive counting numbers.

Positive integers go by many aliases: the counting numbers, the natural numbers, $\mathbb{Z}^+$.

The Consecutive Sums Problem:
Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can?

Experiment, look for patterns, and come up with some conjectures. Write up what you find.

Remember: For now, take only 10 to 15 minutes.

Dissecting the problem

To investigate a problem well, you should get right to its heart. The first two sentences of problem 1 just say what is meant by "sum of consecutive counting numbers," and the last two sentences are merely guidance for the student. The problem's essence is in the middle two sentences:

> Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can?

Even this can be boiled down. The real information is:

> ... counting numbers ... expressed as sum of two or more consecutive counting numbers ...

2. Concealed within that deceptively simple boiled-down version are at least five essential features of the problem. Two are given to you. Find *at least three* others.

(a) It is about a *sum*.

(b)

(c)

(d)

This, of course, is a feature of every problem. Learning to notice what is not *stated is extremely hard for everybody.*

(e) There are restrictions that the problem *could* make, but *does not*. The fact that it *fails* to make more restrictions is part of what makes it *this* problem and not another.

Feature (e) may seem almost too silly to list, but it is important! For example, the problem refers to a "sum of two or more consecutive counting numbers." A more restrictive problem might ask "Which numbers can (or cannot) be expressed as a sum of exactly three consecutive counting numbers?"

This is a new *problem, and an interesting one!*

Similarly, the problem asks which numbers can be expressed at all, in any number of ways. A more specific problem might ask "Which numbers can (or cannot) be expressed in exactly one way (or two or . . .) as a sum of consecutive counting numbers?"

This is another *new and interesting problem!*

Modifying the problem

Two great problems are listed in the previous paragraphs, but they are not the only good ones that come from changing features of the original problem.

How do you decide, before investigating, which will be a worthwhile problem to pursue? Is it intuition? Experience? What goes into your decision?

3. By yourself or with others, brainstorm to see what related problems evolve from this one as you change the features one (or at most two) at a time. Write down and share this set of new problems.

4. Pick one or more of your problems and explore them just long enough to build some preliminary conjectures.

For now, take only 10 to 15 minutes. As before, you won't have enough time for a real investigation, but you should get a rough idea of what the problem has in store.

Problem-posing strategies

Problem 3 asked you to "change the features," but *how* should that be done? Are there any reliable ways to do that and get "good" problems as a result?

As you gain experience, you'll develop your own set of tricks for modifying the features of a problem, but here are four that are almost always among the most useful.

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i. Make a feature more restrictive: If the problem is about triangles, restrict it to right (or scalene or . . .) triangles. If the problem uses a calculation that involves two or more numbers, restrict it to *exactly* two (or three or ...).

This is sometimes referred to as finding special cases*.*

ii. Relax a feature: If the problem is about right triangles, see how it changes if you allow *all* kinds of triangles, or maybe all polygons. If the problem uses a restricted subset of numbers (e.g., only $\{1, 2, 3, \dots\}$), see what happens when you expand that set in various ways.

This is sometimes referred to as generalizing*, or extending the domain.*

iii. Alter the details of a feature: If the problem concerns right triangles, see how it changes if you choose acute triangles. If the problem calls for one set of numbers (e.g., $\{1, 2, 3, \dots\}$), try a different set (e.g., $\{1, 3, 5, 7, \dots\}$ or $\{0, 3, 6, 9, \dots\}$ or $\{0.5, 1, 1.5, 2, \dots\}$). If the problem uses arithmetic operations, see what happens if you systematically alter them (e.g., substituting $+$ and $-$ for $\times$ and $\div$ or vice versa), and if it specifies equality, see what happens if you require a specific inequality (e.g., $>$).

These modifications may change the domain *of a problem, or alter a parameter.*

iv. Check for uniqueness: If the problem only asks *if* something can be done, ask if (or when) it can be done in *only one way*.

Asking how many ways can this be done? *is often productive.*

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5. Apply these and your own rules to generate interesting variants on the following problem:

"How many triangles with perimeter 12 and integer side lengths can you construct?"

You get to be the judge of what is an interesting variant of the problem.

6. Now, go back to the CONSECUTIVE SUMS PROBLEM. Look over the list of features you made for problem 2 and see if applying these rules to each of the features gives you any new problems.

Ways to think about it

The statement of each problem (or a paraphrased version) is provided in the margin for your convenience.

THE CONSECUTIVE SUMS PROBLEM:
Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can? Make a conjecture.

1. Look at a lot of examples, being sure to keep track of everything — maybe make a table. Once you get a tentative conjecture, check it out with a few more examples. Remember that you're not being asked to prove anything. Carefully write down and share your observations and predictions.

Problem: *There are at least five essential features of the consecutive sums problem. One is "It is about a sum." Find* at least three *others.*

2. Analyze the problem statement and context. What "kind" of sum are we talking about here (what's being added)? Is the type or number of addends restricted? How about the relationship between addends?

Problem: *Brainstorm to see what related problems evolve from this one as you change the features one (or at most two) at a time.*

3. First, think of some ways you might alter the original problem to create a new one. One possibility is by adding or removing restrictions. Can you think of other alterations?

Problem: *Pick one or more of the problems you created in problem* 3 *and explore them just long enough to build some preliminary conjectures.*

4. As in problem 1, we're looking for conjectures, not proofs. Try some experimentation and see what you can come up with.

Problem: *Apply these and your own rules to generate interesting variants on the following problem:*
"How many triangles with perimeter 12 and integer side lengths can you construct?"

5. Go through each of the suggested ways of altering problems and see how you might modify this triangle problem.

Problem: *Now, go back to the consecutive sums problem. Look over the list of features you made for problem* 2 *and see if applying these rules to each of the features gives you any new problems.*

6. Did you miss anything when you modified the consecutive sum problem in problem 3?

2. You've got a conjecture. Now what?

In the first section, you investigated a rich problem about consecutive sums and explored ways to modify it and to pose new, related problems. You also worked on exploring a problem beyond its solution. In this section, you'll continue that activity. First, you'll investigate some important—and not so important—connections these modified problems lead to. Second, you'll solve the modified problems, which will lead you to an explanation and proof of the main result (the original consecutive sums problem).

Intuition and mathematical taste

When students are asked to investigate, the ideas and difficulties that arise are inevitably less predictable than when the course is all laid out for them in advance. Some problems they pose lead to unanticipated treasures. Others seem likely to take time without giving the students much in return. Without having more than a few moments to think about the problems, you may find yourself in the position of having to decide which direction to take.

Problems 1–5:
Here are several pairs of variations on the original consecutive sums problem. Look at each pair, and try to decide, *without first pursuing the problems,* which choice seems more likely to lead somewhere.

"Without first pursuing the problems" does not mean that you cannot think about it at all. And "seems more likely to lead somewhere" does not mean "which will surely lead to interesting consequences."

1. (a) Which numbers can be expressed as sums of consecutive prime numbers?
   (b) Which numbers can be expressed as sums of consecutive square numbers?
2. (a) Which numbers can be expressed in exactly seven ways as sums of consecutive counting numbers?
   (b) Which numbers can be expressed in exactly one way as sums of consecutive counting numbers?
3. (a) Which numbers can be expressed as products of consecutive counting numbers?
   (b) Which numbers can be expressed as differences of consecutive counting numbers?
4. (a) Which numbers can be expressed in exactly three ways as the sum of exactly three consecutive odd numbers?
   (b) Which numbers can be expressed as the sum of consecutive odd numbers?

Unit fractions are fractions whose numerators are 1, such as $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, etc.

5. (a) Which numbers can be expressed as the sum of unit fractions with consecutive denominators?
   (b) Which numbers can be expressed as the difference of two unit fractions with consecutive denominators?

Reflect and Discuss

6. How did you make your decision in each of the previous problems? What "rules of thumb" did you use to help distinguish between problems that are probably good and ones that are probably not worthwhile?

And so do his sisters and cousins and aunts and ...

Even good problems can't all be pursued: too many problems, too little time. One way to decide among them is to think about which mathematical connections you most want to make.

Here, reminiscent of the TV game *Jeopardy!*, problems are listed by their answers, because the answers are one way of seeing potential connections to the rest of the curriculum.

PROBLEM

Here is an example of a sister-problem whose answer is on this list: "What numbers are produced as sums of exactly two consecutive counting numbers?"

7. Consecutive Sum Jeopardy:
   In your investigations of the sisters, cousins, and aunts of the Consecutive Sums Problem, what questions (if any) have you run across which have the following sets of numbers as *answers* or partial answers? ⊳
   (a) Only even numbers
   (b) Only odd numbers
   (c) Only prime numbers
   (d) Only powers of two
   (e) Only powers of three
   (f) Only multiples of 3
   (g) Only multiples of 5
   (h) Only square numbers
   (i) Differences of two square numbers
   (j) Only triangular numbers
   (k) Only differences of two triangular numbers
   (l) Only factorials
   (m) Only quotients of two factorials (permutations)

Following up a conjecture: Proof

At the beginning of an investigation, searching for a pattern is often a sensible thing to do. Unfortunately, students' investigations too often *end* when they've found one. Finding and describing an observed pattern is only the first step of an investigation. Next comes the essence of mathematical thinking: the effort to know, and show logically, that the pattern continues, why the pattern occurs, and how it logically follows from and connects with what is already known.

This is your chance to pursue an investigation beyond the first few minutes.

Following are several conjectures people have made as they investigated the CONSECUTIVE SUMS PROBLEM (and its closest cousins). Find a proof or counterexample (an example that shows the statement is not always true) for each one. These 13 problems form a path to a conclusion, so try to justify each statement. If you want some extra guidance, talk to a neighbor or your facilitator, or consult the "Ways to think about it" section beginning on page 11.

Elementary algebra is sufficient for solving these problems, but there are often ways of giving solid proofs—logical arguments, and not just appeals to pattern—that would work for fifth- or sixth- grade students who have never had algebra. See if you can find both *kinds of supports.*

8. The sum of two consecutive counting numbers (CCNs) is always odd.
9. The sum of three CCNs is a multiple of 3.
10. The sum of five CCNs is a multiple of 5.
11. The sum of *any* number of CCNs is a multiple of that number.
12. The sum of any odd number of CCNs is a multiple of that odd number.
13. Odd multiples of 3 (except 3 itself) can always be expressed in at least two ways as sums of CCNs.
14. Odd primes can be expressed in only one way as the sum of CCNs.
15. The sum of an odd number of CCNs can be odd or even. It will be odd if . . .

    Complete, then prove the given statement.
16. The sum of an even number of CCNs can be odd or even. It will be odd if . . .

    Complete, then prove the given statement.
17. The sum of a sequence of CCNs is a multiple of at least one of the numbers in the sequence.
18. The sum of a sequence of CCNs is a multiple of the mean of the first and last numbers in the sequence.

19. Any multiple of an odd number greater than 1 can be expressed as a sum of CCNs.

You may have already guessed where the preceding problems were leading. Even if you haven't, you should now be ready to put it all together.

If there's time, take a few minutes to organize a self-contained proof of the theorem.

20. THE CONSECUTIVE SUM THEOREM
The only numbers which can be expressed as a sum of CCNs are the multiples of an odd number greater than 1. That is, a counting number, N, is the sum of consecutive counting numbers if and only if N is not a power of 2.

Did thinking through and proving your conclusions to the statements in problems 8–19 *help you understand and prove the consecutive sums problem? Are these proofs similar or different from the proofs you encounter (or expect to encounter) in your teaching? If so, how?*

Students often make conjectures that turn out to be false. As their teacher, it is important not only to help them see why the conjecture is false, but also to recognize what *correct* (but perhaps incomplete) observations led them to the false conjecture.

21. For each conjecture that you determined not to be true in problems 8–19, see if you can guess what *correct,* but incomplete, observations might have led to that conjecture.

Ways to think about it

1–5. There are a lot of considerations which might affect your decision of what problem is worth pursuing. Here are some:
- Is the problem *interesting*?
- Are the hypotheses overly restrictive? Not restrictive enough? Just right?
- Can you think of methods of solution (even if you don't yet know the answer) or at least ways of gaining enough information to make an educated guess?

Problem: *Here are several pairs of variations on the original consecutive sums problem. Look at each pair, and try to decide, without first pursuing the problems, which choice seems more likely to lead somewhere.*

6. What were *your* considerations? How did they compare with those of your neighbors? Although it is often difficult to recall the problem-solving process, thoughtful reflection is an important skill to develop. It will help you on the subsequent problems.

Problem: *How did you make your decision in each of the previous problems? What "rules of thumb" did you use to help distinguish between problems that are probably good and ones that are probably not worthwhile?*

7. Think about the investigations and conjectures you made in section 1. Did any of these answers come up then? These might also arise as solutions to your modified problems from section 1. Brainstorm with other participants. If all else fails, think about how these numbers might come up as answers to modified problems, or try some more examples!

CONSECUTIVE SUM JEOPARDY: *In your investigation of the Consecutive Sums Problem, what questions (if any) have you run across which have the following sets of numbers as* answers *or partial answers?*

8. Since the question mentions parity (evenness and oddness), consider the parity of any two consecutive integers: What is the sum of one odd and one even number? Symbolically, what do you know about $n + (n + 1)$?

Prove or disprove: *The sum of two consecutive counting numbers (CCNs) is always odd.*

9. Be careful not to blindly trust examples. Symbolically, such a sum can be expressed in the form $n + (n + 1) + (n + 2)$ or $(n - 1) + n + (n + 1)$. Notice how the second sum is "balanced." Even though both representations lead to a proof, the second might be considered more elegant. Sometimes it's necessary look at a problem in just the right way in order to gain the insight necessary to solve it (or to find a "slick" solution).

Prove or disprove: *The sum of three CCNs is a multiple of* 3.

10. Can you "balance"this sum, too? What is $n+(n+1)+(n+2)+(n+3)+(n+4)$ or $(n-2)+(n-1)+n+(n+1)+(n+2)$?

Prove or disprove: *The sum of five CCNs is a multiple of* 5.

11. Can you balance *every* sum? Another important lesson: sometimes (but not always), the reason it's hard to prove a statement is that it's *false*. If you think that might be the case, look for a counterexample.

Prove or disprove: *The sum of* any *number of CCNs is a multiple of that number.*

Prove or disprove: *The sum of any odd number of CCNs is a multiple of that odd number.*
It might help to recall that we have a formula for the sum of the first n counting numbers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

12. Consider the significance of how many terms are being added. Some questions to consider (you will need to perform some symbolic manipulations to answer them): What does the sum of k consecutive counting numbers "look like"? If the *first* addend is m, the sum starts off as $m + (m + 1) + \cdots$. What is the *last* term? Can you rewrite (i.e., simplify or factor) this sum? Look back at problems 9 and 10 (as well as your earlier work) to determine which multiple you get.

Prove or disprove: *Odd multiples of* 3 *(except for* 3 *itself) can always be expressed in at least two ways as sums of CCNs.*

13. What do you already know about odd numbers as sums of CCNs? (Maybe you ran across this context in session 1.) What about multiples of 3? In problems 8 and 9, you showed that the sum of 2 CCNs is odd and the sum of 3 CCNs is a multiple of 3. However, you probably didn't show that you could get *all* odds and *all* multiples of 3 that way. Can you? Try it!

Prove or disprove: *Odd primes can be expressed in only one way as the sum of CCNs.*

14. How does the result of problem 12 help?

Complete, then prove: *The sum of an odd number of CCNs can be odd or even. It will be odd if ....*

15. Try some examples. When do you get an odd sum? Show that your conclusion always works by using methods you developed for the previous problems.

Complete, then prove: *The sum of an even number of CCNs can be odd or even. It will be odd if ....*

16. Try some examples. When do you get an even sum? Show that your conclusion always works by using methods you developed for the previous problems.

Prove or disprove: *The sum of a sequence of CCNs is a multiple of at least one of the numbers in the sequence.*

17. Be sure you believe it (or not) by looking at some examples. If you're not sure, the examples might give you helpful insight into the problem. And if the statement is false, maybe you'll find a counterexample.

Prove or disprove: *The sum of a sequence of CCNs is a multiple of the mean of the first and last numbers in the sequence.*

18. Try a "balanced" approach. How does the sum of the first and last addends compare to the sum of the second and next to last addends? Are there other equal sums? This can also be done symbolically, but you need to be very careful with the algebra. Does it matter whether the number of addends is even or odd? This affects whether or not there is a middle term in the sequence of CCNs.

Prove or disprove: *Any multiple of an odd number greater than 1 can be expressed as a sum of CCNs.*

19. What is the mean of the first and last addends when there is an odd number of terms? Look at some examples and show it generalizes.

20. First, note that the numbers which are not multiples of an odd number greater than 1 are the powers of 2 (1, 2, 4, 8, 16,...). In order to get an *even* sum, the number of addends must be a multiple of 4. (If you don't see why, refer to problems 12 and 16.)

21. Look back at problems 11 and 17. Can you see how someone might mistakenly think these are true? For instance, for what values of "the number" will the statement of problem 11 be true? Is there a pattern to the types of numbers for which it is true? The conclusion of problem 17 is often true. Try to determine when it works and when it doesn't (try some examples, using a variety of starting points and number of consecutive addends).

THE CONSECUTIVE SUMS THEOREM*: Prove that a counting number, N, is the sum of consecutive counting numbers if and only if N is not a power of* 2.

Problem: *For each conjecture that you determined not to be true in problems* 8–19*, see if you can guess what* correct, *but incomplete, observations might have led to that conjecture.*

3. Do it yourself

During the first two sections of this chapter, you investigated the Consecutive Sums Problem directly and by posing related problems, and you pursued some of the related problems through proof. The goal was to develop both a sense of how to open up a problem and create new ones, and why that is useful, even for the narrowest objective of solving the original problem.

In this section, you will get a chance to apply the same kinds of thinking to one or more new situations, creating, in the process, a whole host of new problems, some ways to investigate them, and probably even some preliminary results.

Two new contexts for investigation

Here are two situations—a problem and a theorem.

For each, list its features and then systematically vary each feature to create new, related problems, as you did in the first two sections. Then select what seem to be the more promising variations and perform a few minutes of exploration just to get a sense of what they might have in store. Then you will choose among the various problems you've worked on and take it further. For now, to exercise your skills at posing new problems that might support an in-depth investigation and get a preliminary sense of where, mathematically, those problems might lead.

Take some time to think and talk about how you might modify these problems. Refer back to section 1 for suggested strategies and don't forget about the *Ways to think about it* section beginning on page 16.

1. The Post Office Problem:
   A particularly quirky post office clerk sells only 7-cent stamps and 9-cent stamps. Can exactly 32 cents' worth of postage be made using these stamps? Can 33 cents be made? Which amounts, if any, cannot be made?

2. The Pythagorean Theorem:
   This theorem, central to an enormous amount of mathematics, can be thought of as a statement about shapes. "The (area of the) square on the hypotenuse of a right triangle is equal to the sum of the (areas of the) squares on the two legs of that triangle."

   Alternatively, the theorem can be thought of algebraically, as a much more generic statement about the way some numbers are related. "The sum of two squares is equal to another square," is often written $a^2 + b^2 = c^2$. Of course, thinking about the theorem completely algebraically ignores what the three letters represent. Without saying either that a triangle whose sides are a, b, and c must be a *right* triangle, and that the sides of any right triangle must always bear this relationship, this is not really the Pythagorean Theorem, but just another arbitrary (if familiar-looking) equation.

The point of view you take in describing or thinking about a theorem will affect the features you choose to describe or alter the theorem.

What's next?

Once you've chosen one of the two problems, analyzed its features, and created some modifications, choose one (or one of its modifications) to investigate more deeply. If there is time available, prove (or disprove and salvage) the conjectures you make.

Ways to think about it

THE POST OFFICE PROBLEM *A particularly quirky post office clerk sells only* 7*-cent stamps and* 9*-cent stamps. Can exactly* 32 *cents' worth of postage be made using these stamps? Can* 33 *cents be made? Which amounts, if any, cannot be made?*

1. Questions to consider when listing or altering the features of the problem:
- What is the question being asked about these sums?
- What are some restrictions it didn't make, but *could*?
- What alternative problems can you think of? What can you *change*? Restrict? Unrestrict?

THE PYTHAGOREAN THEOREM: *This theorem, central to an enormous amount of mathematics, can be thought of as a statement about shapes. "The (area of the) square on the hypotenuse of a right triangle is equal to the sum of the (areas of the) squares on the two legs of that triangle." Alternatively, it can be thought of algebraically, as a much more generic statement about the way some numbers are related. "The sum of two squares is equal to another square," is often written* $a^2 + b^2 = c^2$.

2. Questions to consider:
- What types of triangles are involved?
- What happens if the shapes are changed?
- What happens if other figures were built on the sides of the triangle? Will the relationship between areas be the same?
- Are there triangles for which the side lengths are integers?

4. You know the answer? Prove it.

The next, and perhaps most important, aspect of the problem-solving process is proof. Oftentimes, the rationale given for the necessity of proof is that you don't really know that your solution (or conjecture) is correct until you've proven it. While that may be true in a strict mathematical sense, it's usually the case that you don't sit down to prove something until you *know* that it is true. In classes, for instance, a common assignment is to have students prove that a given statement is true (i.e., that two given triangles are congruent). There seems to be no problem solving, since they are told that the triangles are congruent. For this reason, students often think that proof is something separate from problem solving. So, instead of thinking of proof as necessary to show *that* something is true, let's think of it as helping us understand *why* it is true and giving us insight into what else might be true. In fact, the construction of a proof is a form of problem solving itself.

Learning to write proofs can be a difficult process. One of the reasons for this is that most of us don't have much experience with reading—and understanding—written proofs. Well-written proofs can serve as useful examples for someone who wants to write his or her own proof. Even though you shouldn't think of one proof as a *template* for another, the main ideas of proofs are often the same: Start with the given information, from which you draw conclusions based on what you know, aiming toward the final conclusion, which is what you wanted to prove.

In this session, you will read and critique several different attempts at proving a fact that may have come up in earlier sessions (that the sum of the first n counting numbers is $\frac{n(n+1)}{2}$). Keep an eye out for convincing arguments which leave no doubt that the statements are correct (as well as for arguments that are less convincing). You will also tinker with and modify the alleged proofs. Your tinkering will be another version of the *Variations on a Theme* theme, aiming at modifying these arguments to prove some new facts.

By no means is proof the end *of problem solving. As George Pólya said, "Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. … A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution."*—How to Solve It, 2^{nd} *ed., Princeton University Press,* 1945*, pp.* 14-15

Four proofs for the price of one

There are many ways in which the sum of the first n counting numbers was connected to the Consecutive Sums Problem. Below are four proposed proofs of the conjecture:

If you are not familiar with $\sum$ notation, try to come up with your own definition of this symbol based on what you know about the problem.

$$\textbf{Conjecture: } \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

1. As you read each alleged proof, do the following:

(a) Decide whether the argument is a genuine, acceptable proof. If you feel it is not, fix it.

(b) Rewrite the argument to make it fit a conjecture about the sums of consecutive *odd* numbers starting at 1.

(c) What if the numbers were not consecutive counting numbers but, say, consecutive multiples of 3, or not starting at 1, or ...

You know the game now. What if things were different*? But as you change things, be careful not to destroy the essential elements of the structure of the original proofs. Then your new statements will be new proofs.*

Do you see why this proof attempt is divided into two cases? How does the argument in case 1 fail when n is odd?

Proof Attempt i: We'll consider two cases.

Case 1: Suppose n is even. Then we can rearrange the terms to create pairs like this

$$[1+n]+[2+(n-1)]+[3+(n-2)]+\dots$$

Each of the pairs adds up to $n+1$, and there are $\frac{n}{2}$ pairs. So, in this case, the sum is $(n+1)\frac{n}{2} = \frac{n(n+1)}{2}$.

Case 2: Suppose n is odd. Then $n-1$ is even, so we can use the formula for the sum $T = 1+2+\cdots+(n-1)$, and we get $T = \frac{(n-1)n}{2}$.

$$[1+2+\cdots+(n-1)]+n = T+n = \frac{(n-1)n}{2}+n,$$

which simplifies to $\frac{n(n+1)}{2}$.

Either way, the sum is $\frac{n(n+1)}{2}$. **QED**

QED is an abbreviation for "quod erat demonstrandum," which is Latin for "which was to be demonstrated."

Proof Attempt ii: Define $1 + 2 + \cdots + n = S$. Of course, it doesn't matter what order I add these integers, so it's also true that $n + (n-1) + ... + 1 = S$. Now place these two sums in rows and add "column-wise":

$$\begin{array}{ccccccccccc} & 1 & + & 2 & + & \cdots & + & n-1 & + & n & = S \\ + & n & + & n-1 & + & \cdots & + & 2 & + & 1 & = S \\ \hline & n+1 & + & n+1 & + & \cdots & + & n+1 & + & n+1 & = 2S \end{array}$$

As each of the upper rows is S, the bottom row—their sum—is $2S$. But that bottom row also shows that $2S$ is the sum of n terms, each of which is $(n+1)$.

In other words, $2S = n(n+1)$, so $S = \frac{n(n+1)}{2}$. **QED**

Proof Attempt iii: This attempt tries to use proof by mathematical induction.

We omit algebraic steps in this (and the next) proof. Fill in whatever steps you feel are important.

Suppose you find *some* number for which the conjecture is true. This must be a *particular* number, like 43, for which you can verify the conjecture, perhaps using a calculator.

Let's call that number m.

Even though the number is particular*, we name it* generically *to help us see if the conjecture will generalize to numbers* other *than* 43*. So, in the new statement, we substitute* m*, not* 43*, for* n*.*

$$1 + 2 + \cdots + (m-1) + m = \frac{m(m+1)}{2}.$$

Adding $m+1$ to both sides of this equation, we get

$$1 + 2 + \cdots + (m-1) + m + (m+1) = \frac{m(m+1)}{2} + (m+1).$$

Again, we may think 44*, but we write* $(m+1)$ *so that "the next number" is generic.*

Using algebra to simplify the above equation, we see that

$$1 + 2 + \cdots + (m-1) + m + (m+1) = \frac{(m+1)(m+2)}{2}.$$

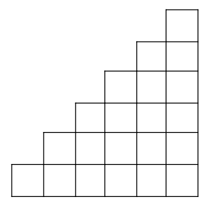
This is what we get if we substitute $m+1$ for n. What this says is that if we already *know* that the conjecture is true for a given m, then we can be certain it is also true for $m+1$.

We know that the conjecture is true for 43—we presumably checked that—so now we know that it must be true for 44. And then 45. And 46. And 47. And so on, forever.

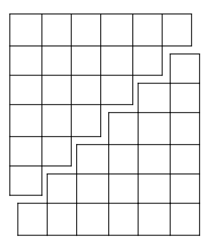
By doing the calculation generically, we have shown that if the conjecture is known to be true for any m*, like* 44*, we know it is true for the next number.*

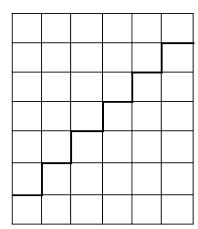
What about 1 through 42? Our proof applies only to numbers *after* the one we checked. So let's start at $n = 1$. The conjecture says that 1 should equal $\frac{1 \cdot 2}{2}$, which it does, so the conjecture is true when $n = 1$ But then (thinking of m as 1) we know that the conjecture must also be true for $n = 2$. But this implies that the conjecture is true when $n = 3$, so it's true when $n = 4$, so it's true when $n = 5 \ldots$. This process will continue forever, so we may conclude that the conjecture for all counting numbers, n. **QED**

Proof Attempt iv: Represent the sum $S = \sum_{k=1}^{n} k$ as the area of a stairstep figure like the one shown below (Pictures are *always* particular. In this one, $n = 6$, but the *argument* works for any n—*imagine* what the figure would look like in the general case).



Now fit a duplicate of the figure on top of the original, as shown below (left), to complete the $n \times (n + 1)$ rectangle shown in the figure on the right.





As in proof 2 above, we see that $2S = n(n + 1)$, so $S = \frac{n(n+1)}{2}$, as conjectured. **QED**

Life after proof

You have developed or encountered several conjectures, including the one that answers the original problem—which numbers can be made?—and you have even proved them. What else is there to a mathematical investigation?

Sometimes, all that is needed is a careful re-organization of what you know into a form that can be presented coherently.

Sometimes, in the course of organizing what you know, you see that the mathematical investigation is really not quite over, and that you *might* want to map the territory more fully, take some side excursions, or fill in some gaps.

How do you choose the direction in which to guide your students? That depends partly on the students and what they've already done, partly on your goals for your class, partly on what else is competing for the time, and partly on the mathematics itself. This fact—the fact that, when students investigate, your own decision making depends partly on the mathematics—places extra demands on you to know not only the central point of the investigation, but all of its cousins and aunts.

Sometimes, putting more time into a problem has such great payoff—either in the mathematical learning or the discipline of pursuing a problem deeply—that it may be worth sacrificing other important things for it. But sometimes there's little more to be gained, and it is time to move on. *Nobody* knows in advance every mathematical problem's cousins and aunts and best-hidden secrets, so it pays to play with the problems a bit—much as you have here—and see what's beneath their surface.

Presenting a mathematical investigation can also be extremely valuable to your students. Partly, it is one contributor to the development of their ability to communicate. Most students will not need to discuss mathematical investigations, per se, for much of their lives, but mathematics, as a technical subject, *is* an ideal place to begin honing the skills of technical communication. And, the fact is that an increasing number of people do have to write and speak about technical matters. A further benefit of learning to do this in mathematics is that math is not *just* a technical subject, but a discipline that is about thinking. Learning to present a mathematical investigation is, in part, learning to express clearly a line of reasoning.

2. Enough of the pep talk. Now it's your turn. You have generated many problems and partial or complete results to some of them (in this section, as well as earlier ones). Pick one, or a group of closely related ones, and organize a presentation of your work to share with other members of your class or workshop (or just for your own intellectual satisfaction).

Ways to think about it

Problem: *As you read each alleged proof, do the following:*
(a) Decide whether the argument is a genuine, acceptable proof. If you feel it is not, fix it.
(b) Rewrite the argument to make it fit a conjecture about the sums of consecutive odd *numbers starting at* 1*.*
(c) What if the numbers were not consecutive counting numbers but, say, consecutive multiples of 3*, or not starting at* 1*, or . . .*

1. In answering part (a), ask yourself whether you understand and agree with each statement. It's OK if you have to ponder some statements or even work out the details to convince yourself that the statements are true. However, it should be relatively straightforward to work them out.

   In part (b), try to maintain the structure of the particular proof you're adapting to the odd consecutive integer case. Which features of the proof will change due to your focus on odd numbers? Which will remain the same? When writing a proof, you need to keep a number of things in mind. One of them is exactly what audience you're writing for. You might not write the same proof for your students that you would write for a college course, for example. Some questions to ask yourself are: *What assumptions can I make about what the readers already know and are able to do? Will the readers of this proof follow each explanation? If not, should I provide more details, or can I assume that the readers will work on it themselves?*

   In part (c), you don't have to answer the specific questions (about multiples of 3 or not starting at 1), although these are perfectly reasonable problems to solve. What are *you* interested in solving?

Problem: *You have generated many problems and partial or complete results to some of them (in this session, as well as earlier ones). Pick one, or a group of closely related ones, and organize a presentation of your work.*

2. Your presentation can include both your methods and your results, but, to feel coherent, its *focus* should probably be on only one or the other: the facts you found or the path you took in finding them. Each has its purpose. You need to decide which story you are telling.
   If you decide to focus on the way you found the facts, the story will probably be richest if it includes the false starts, dead ends, unexpected discoveries, and other *unplanned* elements along with the deliberate strategies and methods. This is not because a confession of one's foibles is good for the soul, but because discovery in real life is rarely a smooth path. Success is often a combination of very good planning, and very opportunistic and intelligent use of serendipity. A focus on results need not tell all this—a clear, logically structured exposition is enough. A story about discovery should tell what the discovery was like.

   Checklist: Have you stated what the driving question was? The related subquestions? Reasoning and proofs? The actual results? The implications or connections?

5. Discerning what *is*; predicting what *might* be

The methods of statistics provide mathematical investigations that are just different enough to justify separate focus in this section. Statistical ideas permeate modern adult life: the news is often presented with statistics of various sorts, and many jobs depend on or communicate through statistical methods. Even middle- and high-school students are increasingly faced with information and representations of that information that cannot be well understood without some grounding in the ideas, methods, and representational techniques of statistics. Without such grounding, people are likely to misunderstand (or entirely miss) reliable and truthful information that they need, and are also vulnerable to being misled by the faulty or fraudulent use of "data." While it's true that "numbers don't lie," it's also true that data can be presented (intentionally and unintentionally) in ways that cause incorrect or incomplete conclusions. This section only barely begins to investigate the nature of statistical investigation. We hope that this brief taste will spur you to further curiosity and investigation on your own.

The technique affects the message

Statistics is often portrayed as a collection of techniques for revealing subtle patterns and true meanings in what would otherwise remain a seemingly inchoate mass of numbers. It cooks the raw data and brings out the flavors.

In fact, like real cooking, the flavor depends not only on the raw ingredients, but also on the cooking process itself. The patterns are not "in" the data, but come from an interaction between the data and the technique used to process those data.

This is not just the old message that improper uses of statistics produce (accidentally or deliberately) misleading results. *All* uses of *any* technique for organizing or massaging data reveal a picture that is influenced partly by the data and partly by the technique. For this reason, it is important to understand the influences that the most basic and common techniques exert—how these do more than "reveal" what is "inherent" in the data.

In this first session, you will pry into the hidden lives of the most elementary statistical processes—visual representation of "unprocessed" data, the three most common measures of central tendency, and the very notion of boiling down a set of data. This hard-hitting exposé will reveal all. You may learn nothing new about how to *perform* these techniques, but you are likely to see something new about their personalities.

Some of the problems you will solve are bad news, mathematically speaking—statistical questions that should never be asked. But because they are paraphrases or subtle variants of commonly encountered problems, the features that make them bad news are important to recognize and understand.

Stem-and-leaf plots and histograms

1. Work through problem **A** below. Also, decide:
(a) which items (if any) seem to require little more than using definitions and procedures (how to read a stem-and-leaf plot, how to compute a mean, and so on);
(b) which items (if any) require some judgment, as well;
(c) which items (if any) are ambiguous or meaningless.

The stem-and-leaf plot takes raw data—in this case, ages—and "plots" them. The top row shows that four people in their thirties entered the store during that 15-minute period: ages 33, 34, 38, and 39. The bottom row records three children under 10.

A. To help decide what kinds of items to keep in stock, a store kept track of the ages of its customers. This stem-and-leaf plot shows the data for one 15-minute period.

3	3 4 8 9
2	5 8 8 8 8 8 9 9 9
1	0 1 1 1 2 2 3 4 4
0	6 7 8

(1) How many people entered the store during that 15 minutes?

(2) Which is the most common age group?

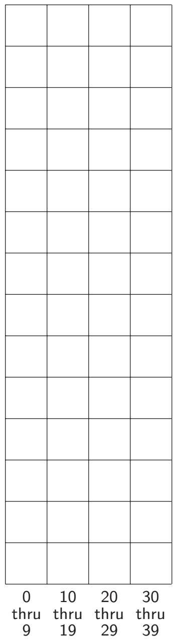
(3) Five customers were the same age. How old were they?

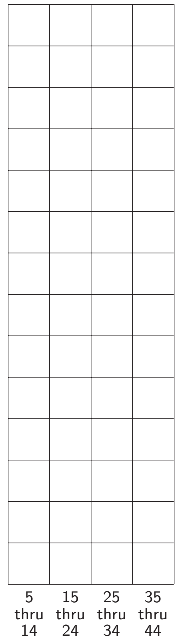
(4) Is 25 a typical age for a customer?

What could be more innocent than these questions? Part **A4** may be a bit ambiguous—it's hard to answer without knowing exactly what's meant by "typical"—but the others seem quite straightforward. Parts **A1** and **A3** test whether a person knows how to read stem-and-leaf plots. This is just convention: Either you know what this representation is intended to mean, or you don't; no real mathematical reasoning is involved.

Part **A2** also seems straightforward, but this time there's something lurking beneath the surface. To see what that is, do the following two problems.

2. Plot the data as histograms in these two ways.





These two graphs are inspired by applying a "what-if-not" (or WIN) strategy to the features of stem-and-leaf plots.

The WIN strategy systematically asks what happens if characteristics of a problem or situation—in this case, a stem-and-leaf plot—are varied. What features of the stem-and-leaf plot are altered in each of these graphs?

Of course, the WIN strategy begins *by asking what* are *the features of a situation. List some characteristics of a stem-and-leaf plot.*

Reflect and Discuss

3. What patterns of customer ages are "revealed" in these two histograms, and what inference might you draw about the clientele of the store based on each pattern? Which corresponds to the stem-and-leaf plot? What, if anything, about the data might you use to help decide which pattern better reflects the Truth about the store's customers?

Stem-and-leaf plots are often used to give a quick sense of the "shape" of the data, so that we can infer what that shape might tell about the data. But because problem **A** gives you only the numbers, and no information about the store to help you interpret those numbers, the question "What might that shape let you infer about the store's clientele?" is a bad question.

Surgeon General's Warning: Interpreting data from the numbers alone may be hazardous to your conclusions. To minimize the risks, try more than one appropriate way to analyze the numbers, and be sure to know as much as possible about the context from which the numbers were drawn.

The main purpose of this problem set is to show you the hazards of using statistical techniques when you have no theory about the context that the statistical techniques are testing.

4. The box below contains another problematic problem. Like in the previous problem, as you work through it, decide:
(a) which items (if any) seem to require nothing more than definitions and procedures;
(b) which items (if any) require some judgment, as well;
(c) which items (if any) are ambiguous or meaningless.

This stem-and-leaf plot might start out like this:

15	*0 2 2*
14	
13	
12	
11	
10	

B. Students recorded their weights to the nearest pound as they tried out for the school's track team. This is the full list: 138, 103, 135, 115, 143, 105, 112, 115, 125, 150, 125, 120, 101, 152, 149, 152, 137, 114, 119, 128, 125, 104, 110, 108, 144, 115, 144, 125, 133, 136, 144, 117, 125, 132.

(1) Make a stem-and-leaf plot to display these weights.

(2) Find the mean, median, mode, and range of these data.

(3) Describe a pattern you see in the data.

5. Again, after working through all parts of problems **B** and **4**, make histograms in the two ways called for below.

150-159								
140-149								
130-139								
120-129								
110-119								
100-109								

Horizontal histogram by decade

145-154								
135-144								
125-134								
115-124								
105-114								
95-104								

Horizontal histogram by rounding

6. What do the histograms "reveal" about the data?

7. What do these experiments reveal about *histograms*?

Parameters of a histogram

Problem 9 *identifies a second feature of histograms, and applies the WIN strategy to vary it.*

As you have seen, stem-and-leaf plots are essentially histograms made in a very particular way. You have seen a couple of trivial variations on the theme of histogram—they can be horizontal or vertical, and the "frequency" can be represented by shaded boxes (the graph-like form) or by digits to the right of a vertical line (the stem-and-leaf form).

You have also seen variations that one might superficially expect to be equally trivial but, on closer inspection, turn out to have consequences that really matter.

8. Problem 5 asked you to make two histograms. A change in one parameter distinguished the two histograms. What was varied?

Interval width—the range of values in any bar of the graph or row of the stem-and-leaf plot—is another feature of histograms. That was kept constant in the four histograms you've just done.

In the plots you've made, the interval width was always 10.

9. Experiment with this parameter *in your head*. When altering this parameter—making intervals narrower or wider—how does it affect the visual pattern you see?

If you have appropriate graphing software, try varying the interval width on a histogram of a large data set. Fathom™ is a particularly intriguing tool to use for this experiment, as it allows you to change the interval width dynamically, and watch the results as you push or pull on the width of the bars. There are also a number of applets available on the Internet. Try a search with keywords `histogram applet`.

Reflect and Discuss

10. Is there a way to determine the "right" interval width for a particular data set?

The Meaning of Mean . . . (and Mode, and Median...)

Problem **B2** (on page 26) asked you to find the mean, median, mode, and range of the data. These four measures of a data set are among the simplest to calculate and understand, but they, too, have properties that are important, yet can be hard to notice.

11. Here are three ways a student might think about the request to "find the mode" in problem **B2**. Each answer is based on a different interpretation of "find the mode." What interpretation leads to each answer? What is *correct* about each interpretation?
  (a) The mode is 125.
  (b) The mode is "110 to 119."
  (c) People's actual weights can't have a mode.

12. Suppose that the weights had been recorded to the nearest tenth of a pound, instead of to the nearest pound. Further suppose that the mean, median, mode, and range were calculated with these new, more accurate data. By how much, *at the very most*, could each of the measures differ from the ones computed with the nearest-pound data?

13. A shoe store owner keeps a record of sizes as each purchase is made and wishes to use this list to help decide what sizes to stock. Which, if any, of the following measures might be most helpful in making this decision? Explain.
(a) mean (b) median
(c) mode (d) range
(e) midrange (halfway between the extremes)

14. For each of the five measures in problem 13, invent a context and question for which that measure is better suited to answering the question than the others. Explain why it's the best.

This comes from Ruma Falk's Understanding Probability and Statistics *(Wellesley, MA: A.K. Peters, 1993). The entire book is a collection of wonderful problems.*

15. Create a set of 8 numbers that will simultaneously satisfy these requirements:
- mean: 10
- median: 9
- mode: 7
- range: 15

16. Eleven servers at a restaurant collect their individual tips in a pocket of their uniforms. At the end of the evening, they dump their pockets into a single bucket and share that money equally. Describe the computation that tells how much each person receives.

17. The eleven servers want a measure of the extent of the inequity of the tipping. What computation might they perform before dumping their pockets into the bucket?

18. The Franklin Knight and Arthur C. Morrow middle schools decided to report the mean score their seventh grade students achieved on a district-wide math test. The mean score of the students who go to Knight was 86; the students who go to Morrow achieved a mean of 82. Under what circumstances could the combined mean of these two schools be anything other than 84? Give a specific example and compute the correct mean for your example.

19. If all you know is that the *median* scores of students at two schools are 86 and 82, what, if anything, can you say about the *median* score of all the students at the two schools?

20. If all of the students in Morrow improve their scores by one point, what happens to the mean score for that school? What happens to the median score?

21. If the top quarter of the students in Morrow all improve their scores by 16 points, what happens to the mean score for that school? What happens to the median score?

Ways to think about it

1. Problem **A** is copied below for your convenience:

> **A.** To help decide what kinds of items to keep in stock, a store kept track of the ages of its customers. This stem-and-leaf plot shows the data for one 15-minute period.
>
> | | |
> |---|---|
> | 3 | 3 4 8 9 |
> | 2 | 5 8 8 8 8 8 9 9 9 |
> | 1 | 0 1 1 1 2 2 3 4 4 |
> | 0 | 6 7 8 |
>
> **(1)** How many people entered the store during that 15 minutes?
> **(2)** Which is the most common age group?
> **(3)** Five customers were the same age. How old were they?
> **(4)** Is 25 a typical age for a customer?

__Problem:__ Work through problem __A__. Also, decide: i. which items seem to require little more than using definitions and procedures; ii. which items require some judgment, as well; iii. which items are ambiguous or meaningless.

Remember, in addition to answering the questions given in problem **A**, you are to analyze the types of questions being asked. What assumptions are you making? Sometimes, these assumptions alter an otherwise ambiguous question–are you making any assumptions that aren't implied by the problem statement or context? Could you make different, but still reasonable, assumptions?

3. Does your answer to the question "What is the most common age group?" depend on the histogram used? Did this surprise you? Which histogram is just a rotation of the stem-and-leaf plot? Does determining which histogram better reflects the "truth" depend on any additional assumptions? If so, what are they? Since we often can't avoid making assumptions, we must at least acknowledge when we do, in order to know whether our conclusions make sense.

__Problem:__ What patterns of customer ages are "revealed" in these two histograms, and what inference might you draw about the clientele of the store based on each pattern? Which corresponds to the stem-and-leaf plot? What, if anything, about the data might you use to help decide which pattern better reflects the Truth about the store's customers?

4. Problem **B** is copied below for your convenience:

> **B.** Students recorded their weights to the nearest pound as they tried out for the school's track team. This is the full list: 138, 103, 135, 115, 143, 105, 112, 115, 125, 150, 125, 120, 101, 152, 149, 152, 137, 114, 119, 128, 125, 104, 110, 108, 144, 115, 144, 125, 133, 136, 144, 117, 125, 132.
> **(1)** Make a stem-and-leaf plot to display these weights.
> **(2)** Find the mean, median, mode, and range of these data.
> **(3)** Describe a pattern you see in the data.

__Problem:__ The box contains another problematic problem. Again, as you work through it, decide: i. which items seem to require nothing more than definitions and procedures; ii. which items require some judgment, as well; iii. which items are ambiguous or meaningless.

Of course, this is very similar to problem 1. In what ways is it different? As before, be aware of your assumptions, and whether or not they are necessary.

Problem: *What do the histograms "reveal" about the data?*

6. Does one of the histograms reveal the "truth" better than the other, or do you need to know something more?

Problem: *What do the histograms reveal about* histograms*?*

7. Are histograms "bias-free," or do the choices you make in setting up the histogram affect the meaning that can be drawn from the histogram?

Problem: *Problem* 5 *asked you to make two histograms. A change in one parameter distinguished the two histograms. What was varied?*

8. One way to answer "What is varied?" is to look for what remains the same. Looking for "invariants" is an important habit of mind to use in mathematical investigation. By learning what doesn't change, you can better find (and focus on) what does change.

Problem: *Experiment with this parameter* in your head*. When altering this parameter—making intervals narrower or wider—how does it affect the visual pattern you see?*

9. Think first about extreme cases. *What if* the intervals were very wide? (How wide is possible?) What is the smallest the interval width can be? What can you say about varying interval width in the intermediate cases? Try a few variations of width and see what changes in the visual representation of the data will result.

Problem: *Is there a way to determine the "right" interval width for a particular data set?*

10. Is there such a thing as the "right" width for a given set of data? If so, what properties would determine whether you found the right width? Be sure to make all of your assumptions explicit. Will the width choice always affect the interpretation?

Problem: *Here are three ways a student might think about the request to "find the mode" in problem* **B2***. Each answer is based on a different interpretation of "find the mode." What interpretation leads to each answer? What is* correct *about each interpretation?*
(a) The mode is 125*.*
(b) The mode is 110 *to* 119*.*
(c) People's actual weights can't have a mode.

11. Try to put yourself in the mind of the student. Alternatively, perhaps you, or someone you know, interpreted the problem in this way. What were you thinking about that led you to the stated conclusion? Remember to look for what is *correct* about each answer.

12. First, determine by what amount each measurement could have varied, then figure out how much of an effect that has on the calculations of mean, median, mode, and range. It might help to consider extreme cases (maximum and minimum) first, but it's important to consider other, intermediate, cases as well.

Problem: *Suppose that the weights had been recorded to the nearest tenth of a pound, instead of to the nearest pound. Further suppose that the mean, median, mode, and range were calculated with these new, more accurate data. By how much,* at the very most*, could each of the measures differ from the ones computed with the nearest-pound data?*

13. In each case, try to imagine what information could be gleaned from the calculation. Is it useful information for the stated context?

Problem: *A shoe store owner keeps a record of sizes as each purchase is made and wishes to use this list to help decide what sizes to stock. Which, if any, of the following measures might be most helpful in making this decision? Explain.*
(a) mean; (b) median;
(c) mode; (d) range;
(e) midrange

14. You'll probably have to choose a context other than the shoe store (or at least needing to know which shoe sizes to keep in stock). Think carefully about exactly what each of the calculations measures in order to find contexts that fit the measure. Don't forget to explain why the measure is best for the stated context.

Problem: *For each of the five measures in problem* 13*, invent a context and question for which that measure is better suited to answering the question than the others. Explain why it's the best.*

15. There are many ways to pursue this. Is it possible to plan out the choice of the numbers in advance, or should you start with some numbers, then change them, if necessary, in order to meet the requirements of the problem. Think about what each of the quantities measures. If your eight numbers have a mean of 11, how can you change one of the numbers so that the set will have a mean of 10? Try experimenting with altering a single number, keeping track of the effect on each of the measures. Is it possible to change a number in order to fix one measure, but leave the other measures unchanged?

Problem: *Create a set of 8 numbers that will simultaneously satisfy these requirements:* mean= 10; median= 9; mode= 7; range= 15.

Problem: *Eleven servers at a restaurant collect their individual tips in a pocket of their uniforms. At the end of the evening, they dump their pockets into a single bucket and share that money equally. Describe the computation that tells how much each person receives.*

16. Start by thinking about how you would informally do this if you were one of the servers. If you introduce notation, be sure to explain the meaning so that your answer is clear to everyone.

Problem: *The eleven servers want a measure of the extent of the inequity of the tipping. What computation might they perform before dumping their pockets into the bucket?*

17. What is meant by "inequity" in this case? Under what circumstances should one server get more (or less) of the tips than another? How can you change the tip distribution so that fairness is restored?

Problem: *The Franklin Knight and Arthur C. Morrow middle schools decided to report the mean score their seventh-grade students achieved on a district-wide math test. The mean score of the students who go to Knight was* 86*; the students who go to Morrow achieved a mean of* 82*. Under what circumstances could the combined mean of these two schools be anything other than* 84*? Give a specific example and compute the correct mean for your example.*

18. Think about this abstractly first, then try some sample data sets. Be careful to avoid unjustified assumptions. For example, you don't have to assume that the scores are distributed systematically. Also, since the two classes are in two different schools, you shouldn't assume that the score distributions are similar, nor should you assume that the two classes have the same number of students. As suggested before, consider some extreme cases.

Problem: *If all you know is that the* median *scores of students at two schools are* 86 *and* 82*, what, if anything, can you say about the* median *score of all the students at the two schools?*

19. Must the median of the two classes' scores be between 82 and 86? Could it be *exactly* 82 or 86? Could it be smaller than 82 or greater than 86? How do you know? Look at some examples, being sure to consider extreme cases, if you're not sure. Once you've got an idea of what the answer is, try to carefully explain the reasons you know your answers are correct.

Problem: *If the top quarter of the students in Morrow all improve their scores by 16 points, what happens to the mean score for that school? What happens to the median score?*

21. Does your answer depend on the results of the other students? Is there a minimum or maximum amount by which the mean will increase, or does that depend on the other students' results, too?