Part 2

Number and Quantity

Number and Quantity

Conceptual Category Overview

Students have studied number from the beginning of their schooling. They start with counting. Kindergarten materials focus on number names, counting, and comparing natural numbers. Early elementary studies use place value, including the concept of zero, and an understanding of place value to begin work with computation. In third grade, fractions are introduced (as recognizing $\frac{1}{b}$ is the representation of one part of a unit partitioned into b equal sized parts) and explored in terms of equivalent fractions and simple comparisons. Throughout early grades, students expand their understanding of number to include more depth of knowledge about fractions and decimals, as well as computation with them.

The progression then moves to integers and irrational numbers by the time students finish eighth grade. Each extension of the set of numbers students study includes examining the "new" numbers to determine which properties still apply, and for example, whether the new numbers (be they fractions, integers, or irrationals) have commutativity, associativity, a distributive property, or identities. Exploring numbers provides students the opportunity to gain a deeper understanding of concepts involving base 10, place value, and computation. For example, students study exponents first as a way of counting and concisely writing a product with repeated factors ($2^3 = 2 \cdot 2 \cdot 2$), but by the end of Grade 8, they include consideration of fractional exponents such as $2^{\frac{1}{2}}$. Further studies will include irrational exponents, irrational numbers such as π and radicals; decimal numbers that do not end but do not have a repetend, for example 1.01001000100001...; logarithms (including natural logarithms) and values of trigonometric functions (though these may include radicals in their calculations).

When students consider quadratic equations, the need for Complex numbers arises. Beginning with equations such as $x^2 = -1$ and continuing to more involved cases (e.g., $x^2 + x + 6 = 0$), students discover and use Imaginary and Complex numbers. Once again, students explore operations and properties with the Complex numbers. The additional standards create an even larger way to consider number quantities and representations. Students may be intrigued by differentiating between algebraic numbers (a number that is the root of some polynomial with integer coefficients) and transcendental numbers (not the root of some polynomial with integer coefficients, such as π), though this is not specifically mentioned in the Common Core.

Students explore matrices and vectors, along with their uses and applications. Teachers should relate transformations in geometry with vector and matrix standards in this domain. Students should build on and use matrix representations as data representations in the statistics conceptual category.

Besides their work with numbers, students also consider Quantity. Labels and measures have been a part of the K–8 standards applied with commonly used concepts such as length, weight, temperature, and speed. Now, students consider modeling situations that require a wide array of measures. Acceleration, dollars per euro, degree-days, and foot-pounds are just a few of the types of measures that may occur. Additionally, students may be involved in modeling situations for which they must create their own measures, for example, gallons per 100 miles traveled when comparing efficiency of cars or persons per television when considering different ways to describe a country's wealth.

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Direct Connections to Number and Quantity in the Middle Grades

As described before, students learning number and quantity in high school build on standards from the middle grades. Teachers provide experiences for Grade 8 students to learn about irrational numbers. Students learn to approximate and compare irrational values to rational numbers. These experiences form a basis for the real number system that students will develop further in high school. Number and quantity in the high school setting expand on the real number system by requiring students to work with rational and irrational numbers in various contexts. Students use properties of rational and irrational numbers to determine the impact of performing operations on these sets of numbers. Students later extend the idea of the real number system to include the complex number system in number and quantity.

SUGGESTED MATERIALS

N.RN	N.Q	N.CN	N.VM	
✓		✓	✓	**CAS** (Computer Algebra System)—A technology capability that computes mathematical expressions symbolically such as a CAS calculator or web-based app
✓		✓	✓	Dynamic graphing technology (i.e., graphing calculators, software)
		✓	✓	Rectangular and polar graphs
			✓	Geoboards
			✓	Applets that relate transformations to vector operations such as http://phet.colorado.edu/sims/vector-addition/vector-addition_en.html from the University of Colorado in Boulder.

NUMBER AND QUANTITY—OVERARCHING KEY VOCABULARY

N.RN	N.Q	N.CN	N.VM	
✓				**Closure** – If an operation is performed on two elements of a set, the result is always an element of the set.
✓		✓		**Complex numbers** – Numbers of the form $a + bi$ where a and b are Real numbers.
			✓	**Matrix** – A rectangular array of numbers. A matrix is defined by its size. The size of a matrix is determined by its number of rows and columns. For a matrix with two rows and three columns, it would be of size 2×3.
✓	✓	✓	✓	**Real numbers** – The set of all possible decimal numbers, that is, the set of all rational and irrational numbers.
			✓	**Vector** – A quantity having direction as well as magnitude. A vector is used to determine the position of one point in space relative to another.

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The Real Number System (N.RN)

Domain Overview

Students use the positive rational numbers in some form as early as third grade. After completing standards for understanding and computing with fractions in sixth grade, students then study integers. A need for numbers other than rational numbers becomes apparent when students learn about the Pythagorean Theorem. Students' knowledge of numbers grow to include irrational numbers and approximations of them. At the high school level, students are able to consider the wide variety of real numbers going beyond their work with square roots and cube roots that arose from geometry (with area and volume explorations). The depth of understanding that there is an infinite number of real numbers between any two given real numbers extends beyond real numbers that solve polynomial equations to include the number *e*, logarithms, values of trigonometric functions, and radian measures and their reliance on π. Here, students work with the properties of exponents to have another way to communicate about irrational numbers (using fractional exponents such as $\sqrt[5]{7^3} = 7^{\frac{3}{5}}$) and to create a deeper conceptual understanding of exponents and their properties that extends beyond counting factors (comparing cases such as 2^3 and $2^{1.33}$).

N.RN—KEY VOCABULARY

N.RN.A	N.RN.B	
✓		**Closure** – If an operation is performed on two elements of a set, the result is always an element of the set.
✓		**Complex numbers** – Numbers of the form $a + bi$ where a and b are Real numbers.
✓		**Imaginary numbers** – A pure imaginary number is a complex number of the form $a + bi$ where $a = 0$. The imaginary unit $I = \sqrt{-1}$.
✓		**Irrational numbers** – Numbers that cannot be expressed as a quotient of two integers and which are not imaginary. The decimal will be non-terminating and non-repeating.
✓		**Rational numbers** – Numbers that can be expressed as a ratio (quotient or fraction) of two integers. All integers are rational numbers since they are expressed as a ratio with a denominator of 1.
✓	✓	**Real numbers** – The set of all possible decimal numbers, that is, the set of all rational and irrational numbers.

Notes

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Number and Quantity | The Real Number System
N.RN.A

Cluster A

Extend the properties of exponents to rational exponents.

STANDARD 1 **N.RN.A.1:** Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{\frac{1}{3}}$ to be the cube root of 5 because we want $\left(5^{\frac{1}{3}}\right)^3 = 5^{\left(\frac{1}{3}\right)3}$ to hold, so $\left(5^{\frac{1}{3}}\right)^3$ must equal 5.

STANDARD 2 **N.RN.A.1:** Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Cluster A: Extend the properties of exponents to rational exponents.

Students build on their work with integer exponents to consider exponents that are not integers (e.g., $5^{\frac{1}{2}}$, $2^{-\frac{2}{3}}$ and $6^{2.3}$). By using calculations, such as $\left(7^{\frac{1}{3}}\right)^3 = 7^{\left(\frac{1}{3}\right)3}$, students create the meaning of fractional exponents and are then able to rewrite radical and exponential expressions in order to solve problems and simplify them. Procedural fluency between using radical and exponential notation follows from the understanding of both forms.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 5. Use appropriate tools strategically.
SFMP 7. Look for and make use of structure.

The structure of exponential rules is used to make sense of rational exponents. Students use rational exponents and radicals in problem solving. CAS is a good tool for exploring situations with rational exponents, such as comparing decimal approximations of $2^{\sqrt{2}}$ and $2^{1.4}$ to form conjectures about betweenness and size or $2^{\sqrt{2}}$, $\sqrt{2}$, and $2^{\frac{1}{2}}$ or graphing functions related to different forms of an expression, such as $f(x) = \sqrt{x}$, $g(x) = x^{0.5}$ and $h(x) = x^{\frac{1}{2}}$.

Related Content Standards

A.SSE.A.1 A.SSE.A.3 A.CED.4 8.NS.A

Notes

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STANDARD 1 (N.RN.A.1)

Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{\frac{1}{3}}$ to be the cube root of 5 because we want $(5^{\frac{1}{3}})^3 = 5^{(\frac{1}{3})3}$ to hold, so $(5^{\frac{1}{3}})^3$ must equal 5.

Students use the exponent property $\left(a^b\right)^c = a^{bc}$ from earlier grades to derive the meaning of rational exponents. The connection between inverse operations (such as multiplication undoes division) is expanded to radicals and exponents. For example, $\left(\sqrt{25}\right)^2 = 5^2 = 25$ is a good starting point for considering how a square root and squaring undo each other. Exploring similar problems by hand and with technology resources leads students to discover what exponent must stand for a given radical (that $\frac{1}{2}$ is the exponent that means the same thing as $\sqrt{\ }$). Problems such as $\left(\sqrt{25}\right)^3 = 25^{\frac{3}{2}} = \left(25^{\frac{1}{2}}\right)^3 = 5^3 = 125$ further extend the meaning of fractional exponents beyond fractions with a numerator of one. Students must also connect the meaning of functions with the restricted range of operations such as the square root—that is, students will be able to explain that $\sqrt{9} = 3$ and not $\sqrt{9} = \pm 3$ in order to satisfy the demand that a square root be a function.

What the TEACHER does:

- Uses problems such as $\left(\sqrt{3}\right)^2$ and $\sqrt{(3^2)}$ to consider rational exponents.
- Provides problems that give a context for using rational exponents (e.g., "What radius gives a sphere a volume of 4π cm³?").
- Presses students to explain the meaning of fractional exponents in different contexts.
- Provides problems that enable students to determine the difference between computing a value such as $\sqrt{4}$ and finding the solution set of a related equation, such as $x^2 = 4$.

What the STUDENTS do:

- Explain the meaning of rational exponents using examples such as

  $x^3 = 25$

  $\sqrt[3]{x^3} = \sqrt[3]{5^2}$

  $(x^3)^{\frac{1}{3}} = (5^2)^{\frac{1}{3}}$

  $x^{3\times\frac{1}{3}} = 5^{2\times\frac{1}{3}}$

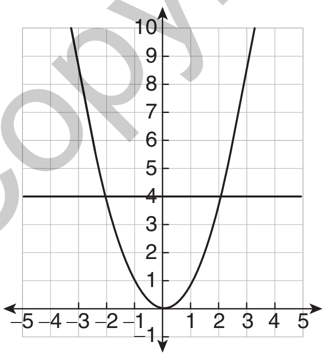
  $x^1 = 5^{\frac{2}{3}}$

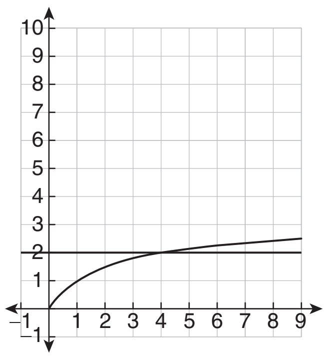
  $x = 5^{\frac{2}{3}}$

- Define the principal square root function and describe the difference between $\sqrt{4}$ and the solution set of $x^2 = 4$.

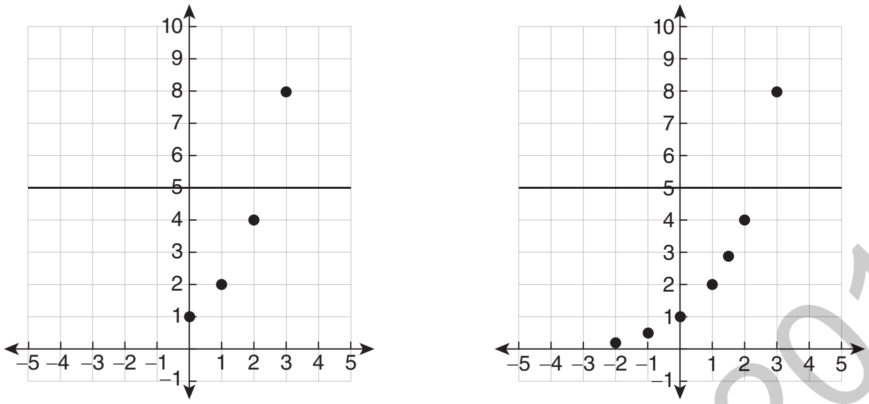
Addressing Student Misconceptions and Common Errors

Students may confuse the concepts of additive inverses and multiplicative inverses when working with rational exponents. Provide problems that allow students to differentiate between the two, such as $\left(4^{-2}\right)^2$ and $\left(4^{\frac{1}{2}}\right)^2$. The figure below on the left shows the graph of $y = x^2$ and its intersection with the line $y = 4$, showing there are two solutions. The figure on the right shows the graph of $y = \sqrt{x}$ intersecting with $y = 4$, which has only one solution.





Students frequently confuse taking the square root of a number with finding the solution set of an equation, such as $x^2 = a^2$ for $a > 0$ so that they give two solutions for $\sqrt{a^2}$. Practice with graphs, the definition of functions, and technology can address this misconception. Calculators can help students explore different expressions graphically (such as solving $x^2 = 4$ by graphing $g(x) = x^2$ and $h(x) = 4$ and finding the intersections as compared with graphing $f(x) = \sqrt{x}$ and $x = 4$ with only one solution) to recognize the need for a "principal" square root function. Though students may have graphed exponential functions as continuous, up to this point, they have generally considered the functions discretely. Rational exponents "fill in" some of the points between the integer exponent values on a graph—that is, for $g(x) = 2^x$, between 2^1 and 2^2, there are now values that students can calculate and explain, such as $2^{\frac{3}{2}}$. Below, the figure on the left is the graph of $g(x) = 2^x$ with only whole-number exponents while that on the right has negative and some fractional exponents.



In the first graph, only whole numbers are used. In the second graph, using decimal and fractional exponents starts to make the graph look more like the continuous function using all real numbers as the domain will allow. Students can try negative exponents, such as here with –1 and –2, and fractional exponents, such as $\frac{3}{2}$.

The additional step of considering irrational exponents to explain continuity is still needed. The concept of irrational exponents follows from an understanding of decimal exponents, such as $2^{3.14}$ is approximately equal to 2^{π}. Students can investigate $2^{1.1}$, $2^{1.01}$, $2.^{1001001}$, and more to be able to consider and compare their values. Additional studies of irrational numbers are possible here. Ask students, "Is 1.001000100001 . . . able to be written as a root, as far as you know? Is it possible for 1.001000100001 . . . to be an exponent? What does that mean?" The concept of an infinite number of numbers being between any two numbers is both fascinating and difficult to truly understand and can be investigated with numerical cases and graphs.

Related Content Standards

A.SSE.A.1 F.IF.C.7e

Notes

STANDARD 2 (N.RN.A.2)

Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Students are able to use both radical and exponential forms to write expressions and can translate flexibly between them. Students use symbolic examples, such as $a^2\sqrt{a} = a^2 \cdot a^{\frac{1}{2}} = a^{\frac{5}{2}}$, and contextual examples, like solving $V = \frac{4}{3}\pi r^3$ for r.

What the TEACHER does:

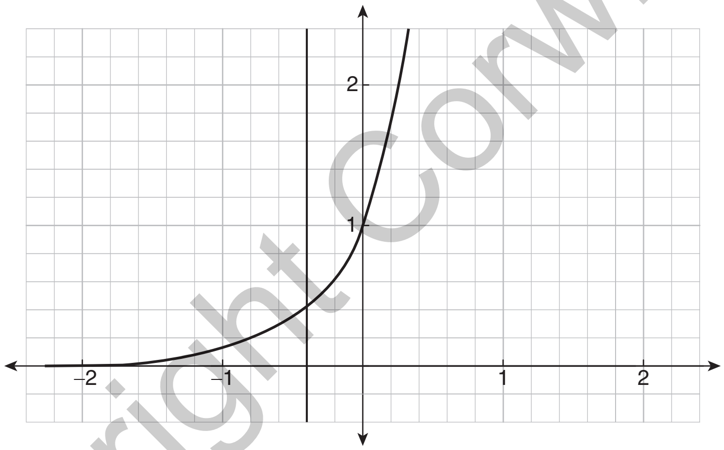
- Uses problems that allow students to use either radical or exponential forms and requires them to explain their reasoning for their choice.
- Solves contextual problems such as solving the volume of a cube for one side, $V = s^3$, $s = V^{\frac{1}{3}} = \sqrt[3]{V}$.
- Requires students to discuss the meanings of their computations when rewriting and simplifying radical and rational exponent expressions.

What the STUDENTS do:

- Explain the meaning of rational exponents in terms of radicals and roots.
- Translate fluently between radical and exponential forms.
- Explain their reasoning when using either notation to solve problems involving radicals.

Addressing Student Misconceptions and Common Errors

Negative exponents can be a problem when using fractional exponents. Students often think $9^{-\frac{1}{2}}$ means –3 instead of $\frac{1}{3}$. Using a calculator to calculate $9^{\frac{-1}{2}}$ helps, as does looking at the graph of $y = 9^x$ and $x = -\frac{1}{2}$, to see where the functional value occurs.



The curve is $y = 9^x$, and the vertical line is $x = -\frac{1}{2}$. The scale shows the intersection of the curve and graph is a positive number that is between zero and 0.4, so –3 is excluded as a solution while $\frac{1}{3}$ appears as a viable estimate of the intersection value.

Connections to Modeling

Solving problems that involve formulas with exponents and/or radicals. Solving problems that involve volume and area.

Related Content Standards

A.SSE.B.3 F.IF.C.7e

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Number and Quantity | The Real Number System
N.RN.B

Cluster B

Use properties of rational and irrational numbers.

STANDARD 3

N.RN.B.3: Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Cluster B: Use properties of rational and irrational numbers.

Students connect the properties of closure for addition and multiplication of integers (and its subsets) to rational and irrational computation. Students consider problems such as $(2+\sqrt{6})+(1-\sqrt{3})$, $(2+\sqrt{6})-(1-\sqrt{3})$, and $(2+\sqrt{6})(1-\sqrt{3})$ looking for patterns, commonalities, and differences to create an understanding of arithmetic with rational and irrational numbers.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 5. Use appropriate tools strategically.
SFMP 7. Look for and make use of structure.

Students will be examining the underlying structure of number arithmetic and applying it to the real numbers. Teachers should use tasks that encourage problem solving so students may discover properties in this cluster. Teachers should expect students to make sense of the problems and stay on track when solving them to discover the properties and learn the underlying mathematics. Students may use graphs, paper-and-pencil calculations, graphing calculators, and CAS to explore properties in this cluster. Students are specifically required to explain properties of computations with rational and irrational numbers.

Related Content Standards:
A.SSE.2 A.SSE.3 A.APR.4 8.NS.A

Notes

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STANDARD 3 (N.RN.B.3)

Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Students work with symbolic forms of rational and irrational numbers to make conjectures about closure for addition and multiplication and to be able to explain why their conjectures work. The key word is *explain*. It is not sufficient for students to apply an algorithm to complete a calculation; they must understand what that calculation means.

What the TEACHER does:

- Has students explore different computations with only rationals to get a feel for what it means for a computation to have a value that either is or is not part of the set being used. Expects students to use tools (calculators or by-hand computation) appropriately.
  Example: The question, "Is the sum of two unit fractions always a unit fraction?" sets the stage for students to explore with irrational numbers by first considering numbers with which they are already familiar. This also is an instance where the question is resolved by use of a counter-example:
  $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, which has a sum that is not a unit fraction.
  Students can then look at the product of two rational numbers and begin to explore.
- Has students explore different computations with irrationals, comparing calculations such as $2+\sqrt{3}$, $2+3\sqrt{2}$, and $2+\sqrt[3]{2}$ with decimal approximations, symbols, and CAS.
- Has students explore different computations with both rational and irrational numbers such as the product of 2 and $2+3\sqrt{2}$.

What the STUDENTS do:

- Define closure with an operation, and apply closure to the addition of two rationals and two irrationals and multiplication of two rationals.
- Explore whether closure applies when multiplying two irrational numbers, such as $\sqrt{2}$ times $\sqrt{8}$.
- Supply examples and counter-examples of properties.

Addressing Student Misconceptions and Common Errors

Students can explore many different calculations with CAS. For example, when $\sqrt{2}-\sqrt[3]{2}$ is put into Wolfram Alpha, it returns $\sqrt{2}-\sqrt[3]{2}$ for the "exact result" and 0.15429251247822 . . . (with over 50 places), which goes beyond the standard but allows students to see the strengths and weaknesses of CAS.

Students may note that the sum of two irrational numbers is not a rational number by looking at the symbolic expression or the non-terminating, non-repeating decimal. Students may conjecture that the sum of two irrational numbers is irrational. It is important for students to realize an example is not an explanation or a proof of a property.

Explaining that the sum of two rational numbers is always rational will generally start with students using examples. This is not an explanation. Ask questions that require students to explain what a rational number is (such as saying a rational number is the quotient of two integers), and then, ask students to try to create a counter-example. Classroom discussion should focus on closure for addition, subtraction, and multiplication of integers and how that relates to adding rational numbers $\frac{2}{3}+\frac{5}{8}=\frac{16}{24}+\frac{15}{24}=\frac{16+15}{24}=\frac{31}{24}$. First, students use multiplication of integers to get equivalent forms of the fractions with common denominators. Then, they use addition of integers to get the numerator. The solution is still an integer divided by an integer. Students may want to consider the mixed number but should be asked to think about whether that fits the form the property refers to. Similar explorations with all of the properties in the standard are helpful.

Ensure students know what closure means by working with integers and subsets of integers with addition, subtraction, multiplication, and division. Students may have difficulty with the arithmetic of rationals and irrationals. CAS and calculators may assist when students are exploring.

Related Content Standards

A.SSE.A.2 A.SSE.B.3 A.APR.A.1

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Sample PLANNING PAGE

Number and Quantity

Domain: The Real Number System

Cluster B: Use properties of rational and irrational numbers.

Standard:

N.RN.B.3: Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Standards for Mathematical Practice:

SFMP 1. Make sense of problems and persevere in solving them.

Students explore patterns in computations to discover how closure works (or doesn't work) for addition and multiplication for different computations with rational and irrational numbers.

SFMP 3. Construct viable arguments and critique the reasoning of others.

Students share and explain their rules and offer justifications for them. Though the word *explain* is not at the level of proof, students understand that examples of computation are not a sufficient as an explanation.

SFMP 5. Use appropriate tools strategically.

Students choose between by-hand computation and technologically assisted computation in testing cases and making conjecture.

SFMP 7. Look for and make use of structure.

Students connect the structure of the mathematical property of closure as it applies to the cases of addition and multiplication with rational numbers.

Goal:

Students practice calculations with rational and irrational numbers to make generalizations that are the basis of explanations as to why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Planning:

Materials: Students need a copy of the prompts for The Real Number System (**Reproducible 2**).

Sample Activity: The students are given the following tasks. Initial work is done individually for 10–15 minutes. Then, students pair with another student to share discoveries and work. The class closes with a whole-class discussion where explanations are shared and discussed.

1. Is the sum or product of two rational numbers always rational? Why, or why not? Provide examples and an explanation.
2. Is the sum of a rational number and an irrational number rational or irrational? Provide examples and an explanation.
3. Is the product of a nonzero rational number and an irrational number rational or irrational? Provide examples and an explanation.

Sample PLANNING PAGE (Continued)

Questions/Prompts:

I see you have three examples, and then, you declare the sum of two rational numbers is always rational. Do you know that's always true from just your examples? Why?

When testing adding a rational and irrational number, you added $2.3 + \pi$ and said the sum was 5.44. Which of your numbers was irrational? (You're expecting π for an answer.) Why did you write π as 3.14? Is 3.14 irrational?

When you use your calculator to compute 2π, are you convinced the product is irrational? Why?

How can you extend your use of examples to make an explanation that allows you to state "always" when discussing the prompts?

Differentiating Instruction:

The use of appropriate questions and strategic use of sample problems to give students an initial step are important ways to differentiate instruction. Students may need to be prompted for examples of irrational numbers besides π or a square root, so the teacher might ask, "What do you know about irrational numbers? How can that help you write an irrational number and a rational number so you may consider whether the sum of a rational and an irrational number is irrational or not?" Similar questions that assess understanding but that do not give a direct path to a solution are essential to ensuring the task remains at a higher cognitive demand than would occur if students were just asked to complete a set of suggested computations and then make a generalization.

Struggling Students:

Suggest one sample computation, such as $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, and ask what that result implies. Encourage the students to try similar problems. Do the same type of questioning with the other cases. The use of technology can assist with the computations so students may concentrate on the patterns they are seeing and attempt to make a generalization.

Extensions:

Students explore the product of an irrational number and an irrational number to make conclusions about whether the product is always, sometimes, or never irrational. The students explain their decisions with a logical argument and/or the use of counter-examples.

Notes

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Reflection Questions: The Real Number System

1. Discuss operations with even and odd integers. How can this be used to build student understanding of closure? How can you transition from the understanding of closure in these cases to rational number addition and multiplication?

2. N.RN.A.1 requires students to explain how the definition of rational exponents follows from extending the properties of integer exponents to those values. How does the principal square root relate to the following false statement?

$$2 = \sqrt{4} = \left(2^2\right)^{\frac{1}{2}} = \left[(-2)^2\right]^{\frac{1}{2}} = (-2)^1 = -2$$

3. What does it mean to a student to explain a property? Will you expect your students to be able to use a symbolic explanation (such as $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ and definitions)?