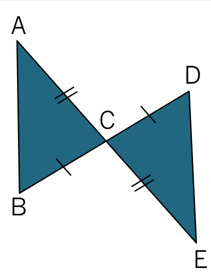
Preface

What do you think of when you hear the word *proof*? It may bring to mind a set of statements and reasons in two columns that you encountered in high school geometry (e.g., see Figure P.1) or perhaps a number theory proof by mathematical induction from a college course (e.g., see Figure P.2). Perhaps when you hear the word *proof* you think of something that is hard to do, or a topic not appropriate for most secondary students. If you have any of these thoughts about proof, you are not alone. For example, in a group of 192 pre-

FIGURE P.1 Geometric proof in two columns.



Given: $AC = CE$
$BC = CE$

Prove: $\Delta ABC \cong \Delta EDC$

Statements	Reasons
1. $AC = CE$ $BC = CE$	**1.** Given
2. $\measuredangle ACB \cong \measuredangle DCE$	**2.** Vertical angles are congruent.
3. $\Delta ABC \cong \Delta DEC$	**3.** SAS: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the triangles are congruent.

FIGURE P.2 Number theory proof by mathematical induction.

Problem: For any natural number n, $n^3 + 2n$ is divisible by 3.

Proof:

Basis Step: If $n = 0$, then $n^3 + 2n = 0^3 + 2*0 = 0$. So it is divisible by 3.

Induction: Assume that for an arbitrary natural number n, $n^3 + 2n$ is divisible by 3. -------- *Induction Hypothesis*

To prove this for $n + 1$, first try to express $(n + 1)^3 + 2(n + 1)$ in terms of $n^3 + 2n$ and use the induction hypothesis.

$$\begin{aligned}(n + 1)^3 + 2(n + 1) &= (n^3 + 3n^2 + 3n + 1) + (2n + 2) \\ &= (n^3 + 2n) + (3n^2 + 3n + 3) \\ &= (n^3 + 2n) + 3(n^2 + n + 1)\end{aligned}$$

which is divisible by 3, because $(n^3 + 2n)$ is divisible by 3 by the induction hypothesis.

End of Proof.

Source: http://www.cs.odu.edu/~cs381/cs381content/induction/example5/example5.html

service mathematics teachers, 18% had negative opinions toward mathematical proof and provided responses that were classified as “proof is hard to understand,” “proof is unnecessary,” and “proof is the nightmare of the students” (Ersen, 2016). Knuth (2002b) found that the secondary teachers he interviewed saw proof as “an appropriate idea only for those students enrolled in advanced mathematics classes and for those students who will most likely be pursuing mathematics-related majors in college” (pp. 73–74).

The purpose of this book is to dispel some of the common conceptions regarding what proof is, how it can be written, and when and for whom it is appropriate. Alan Schoenfeld, a prominent mathematics educator, stated,

> Proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels. (1994, p. 76)

We agree with Schoenfeld’s claim, and in the chapters that follow, we will provide guidance about how to make reasoning-and-proving a reality in your classroom.

WHAT THIS BOOK IS ABOUT

The content of this book is based on the importance of mathematical reasoning-and-proving in middle and high school mathematics—processes that can and do enhance students' understandings of the math that they are learning and help them to retain those understandings over time. Much of what we know from over 40 years of careful and rigorous research about learning and understanding mathematics is that kids of all ages benefit in many ways when they are required to make sense of the mathematics they are learning—when they engage in sense-making as a regular activity in math class. We believe that a powerful way to engage students in sense-making is to engage them in reasoning-and-proving activities.

What does that kind of instruction look like? How do you teach in a way that supports reasoning-and-proving in your classroom? This book will help to enhance your own understanding of reasoning-and-proving, and provide practical teaching practices to help your students reason-and-prove in your middle and high school mathematics classrooms.

Why learn more about how to incorporate reasoning-and-proving into your mathematics instruction? One reason is that the field of mathematics education has long emphasized the importance of reasoning and sense-making in K–12 mathematics for the learning of mathematics with understanding. A common thread that has run through standards and policy documents (e.g., NCTM, 1989, 1991, 1995, 2000) over the past three decades, as well as in professional publications about the teaching of mathematics (e.g., NCTM, 2006, 2009, 2014), is that they advocate for the necessity of learning environments that encourage and support mathematical reasoning-and-proving. This thread continued with the publication of the *Common Core State Standards–Mathematics* (CCSSM) (National Governors Association & Council of Chief State School Officers, 2010), which also advocates for these types of learning environments through articulating the eight Standards for Mathematical Practice (more on those in Chapter 1).

But even with repeated calls for incorporating reasoning and sense-making into K–12 mathematics learning environments, research has documented that students' opportunities to engage in environments that encourage reasoning, sense-making, and proving have been quite limited (Horizon Research, Inc., 2013; Weiss, Pasley, Smith, Banilower, & Heck, 2003). If we want our students to be able to make sense of the mathematics they are learning, then we believe that incorporating reasoning-and-proving activities into middle and high school mathematics has the potential for strongly supporting students' understanding of important mathematics.

One of the challenges of realizing the vision of reasoning-and-proving set forth in the documents referenced above is that many mathematics teachers have not had the opportunity to engage in reasoning-and-proving activities as mathematics learners or to consider how to incorporate reasoning-and-proving into their mathematics teaching. The vast majority of us never learned mathematics in this way, making it difficult to envision the kinds of mathematics learning environments that support our students to be mathematical thinkers.

If you are one of those teachers, this book is for you.

THE PROJECT BEHIND THIS BOOK

As a group of former mathematics teachers and current university-level mathematics teacher educators, we were concerned about the state of reasoning-and-proving that had been documented by research conducted in U.S. mathematics classrooms. As a result of this concern, we worked together to develop and propose a research and materials development project that focused on helping mathematics teachers learn more about reasoning-and-proving and how these processes play out in classrooms. As a result of this proposal, the National Science Foundation (NSF) awarded us a grant that supported the development of the contents of this book under the auspices of a project titled *Cases of Reasoning-and-Proving (CORP) in Secondary Mathematics* (NSF #DRL 0732798). Fran Arbaugh and Margaret (Peg) Smith, the first two authors of this book, served as co-directors of the project with the remaining co-authors, Justin Boyle, Gabriel J. Stylianides, and Michael Steele, helping to develop the materials. Other mathematics educators contributed to the development of the materials in different ways, and we express appreciation for their contributions in the book's acknowledgements.

During development, we piloted the CORP materials with both preservice and practicing middle and high school mathematics teachers and revised them based on what we learned from those pilots. So, the learning activities that form the core of this book are tried and tested with hundreds of teachers. More formal evaluation of the materials (e.g., Boyle, 2012; Karunakaran, Freeburn, Konuk, & Arbaugh, 2014) has confirmed that teachers learned a great deal about reasoning-and-proving from interacting with the activities. In short, we have found that our approach for learning about reasoning-and-proving, as well as how to implement reasoning-and-proving activities in mathematics classrooms, is effective. We now want to give more teachers access to the opportunity to learn about reasoning-and-proving through the publication of this book.

HOW TO USE THIS BOOK

The primary audience of this book is practicing middle and high school mathematics teachers and those who are studying to be middle and high school mathematics teachers. Teachers can learn from the book by engaging with it in a number of different ways:

- **Individuals** can use it for independent study, working through the book on their own.
- **Small groups of teachers** can use this book as the basis for a book study in a professional learning community, working through the book together and engaging in discussions about the content.
- **Professional developers** can use this book to plan and implement professional development focused on enhancing middle and high school teachers' knowledge and teaching practices focused on reasoning-and-proving.
- **Mathematics teacher educators** can use this book in planning and implementing university-level coursework in mathematics education.

We wrote this book in a way that encourages readers to engage actively in learning activities about reasoning-and-proving because we believe that the best learning occurs through doing and reflecting. Smith (2001) described engagement in such activities as "practice-based professional development." We believe that simply reading through the book without doing the activities will limit what you learn. In other words, you will get the most out of the book if you **stop and engage** as you work through the book.

Across the chapters, there are mathematical tasks for you to do, sets of student work for you to analyze, narrative cases of mathematics classrooms for you to examine and reflect upon, and "Pause and Consider" prompts to support you in reflecting on what you are learning. We based the narrative cases on events that occurred in real mathematics classrooms, and while not written verbatim from classroom audio- and video-recordings, we have represented with fidelity the heart of what happened in the classrooms. The examples of student work are from real students, many of which have been rewritten for the purpose of making them more readable.

We have clearly indicated those learning activities and reflective prompts within the chapters and have included some blank pages in the back of the book for you to use for recording your reflections or making teaching notes along the way. It is likely, though, that you will need a separate notebook where you can complete the mathematical tasks that are part of many activities. The narrative cases are included in the book as appendixes, and there are several documents on the companion website that you can download and use along the way—look for the website indicator in the margin to help you know what is available on the website.

The book is organized into seven chapters, and the content builds from chapter to chapter—the activities in later chapters depend on you having worked through activities in earlier chapters. So, it is important that you work through the chapters sequentially. Here is a brief preview of each chapter:

- In **Chapter 1: Setting the Stage**, you will reflect on your current conceptions of proof and the role of proof in middle and high school mathematics and read more about the background for the book. We also provide information about our conceptions of reasoning-and-proving that undergird the contents of the book, including why we use a hyphenated version of the phrase.
- In **Chapter 2: Convincing Students That Proof Matters**, you will engage in a series of mathematical tasks that you can use with your students to convince them that the use of numerous examples is not enough to prove a mathematical conjecture. Then you will consider the implementation of the series of tasks in two different mathematics classrooms through reading and analyzing narrative cases.
- In **Chapter 3: Exploring the Nature of Reasoning-and-Proving**, you will develop a set of criteria that can be used to judge when a mathematical argument counts as proof. Then

you will learn about a framework for reasoning-and-proving that Gabriel J. Stylianides (one of the authors of this book) developed and consider how that framework can be used to organize reasoning-and-proving activities in your classroom.

- In **Chapter 4: Helping Students Develop the Capacity to Reason-and-Prove**, you will consider four effective teaching practices that you can use in your classroom to support your students' engagement in reasoning-and-proving activities. Then you will visit two mathematics classrooms, through narrative cases, and analyze how those teaching practices are implemented with students.
- In **Chapter 5: Modifying Tasks to Increase Reasoning-and-Proving Potential**, you will learn about strategies for modifying tasks from curriculum materials so that they better support students to engage in reasoning-and-proving. In addition, you will consider how establishing clear learning goals supports implementation of reasoning-and-proving activities with your students.
- In **Chapter 6: Using Context to Engage in Reasoning-and-Proving,** you will focus on how context supports students to reason-and-prove as you solve a reasoning-and-proving task, analyze a set of student responses to that task, and then analyze two narrative cases captured in mathematics classrooms where the task was implemented with high school students.
- In **Chapter 7: Pulling It All Together**, you will reflect back on the content of the book, considering key ideas presented in Chapters 1–6 and important tools that support engaging in reasoning-and-proving as well as the teaching of reasoning-and-proving.

We hope that this book provides a valuable learning opportunity for you. You will certainly broaden and deepen your current understandings of reasoning-and-proving *and* how to create learning environments that help students to enhance their capacities to reason-and-prove.

Fran Arbaugh
Margaret (Peg) Smith
Justin Boyle
Gabriel J. Stylianides
Michael Steele